Constructing Trig Values: 
The Golden Triangle and the Mathematical Magic of the Pentagram

A Play in Five Acts

"Mathematicians always strive to confuse their audiences; where there is no confusion there is no prestige. Mathematics is prestidigitation." From Mathematics made difficult by Carl E Linderholm.

Prologue: “You Can with Beakman and Jax,”

In the spring of 2006 I was assigned to teach a trigonometry class for the first time since 1992. In preparation for the course I made the following “rather standard” chart showing the well known “exact” values of cosine and sine around the unit circle. For the values not “on axis” (i.e., not 0°, 90°, 180°, 360°), the results are easily derived from either an equilateral triangle or an isosceles right triangle.

The Monday after Christmas, I was doing some stretching while reading the Sunday comics. In the “science” strip, “You Can with Beakman and Jax”, there was a discussion of the “magic” of the five pointed “Christmas star” and the occurrence therein of the “Golden Ratio”. I must admit with some embarrassment that this was all news to me. I had never really thought much about pentagrams before. But I immediately realized that this meant there was another group of “exact” trig values based on a 18°, 72° right triangle. This presentation is based on my “re-discovery” of these very well known properties.
Act I: “The Golden Ratio”

In Book VI of the *Elements* Euclid divides a line segment into two intervals in such a way that the length of the longer interval to the length of the shorter interval is equal to the length of the whole interval to the length of the long interval.

Designate the ratio as the Greek letter phi. Then, \( \phi = \frac{AB}{BC} = \frac{AC}{AB} = \frac{AB + BC}{AB} = 1 + \frac{1}{\phi} \). Hence,

\[ \phi^2 = \phi + 1 \]

or equivalently,

\[ \phi^2 - \phi - 1 = 0 \]

Since this ratio is positive, the quadratic formula gives the result

\[ \phi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2} = 1.6180339887498948482045868343656\ldots \]

Euclid called this number the extreme and mean ratio. Later it acquired the name the “Golden Ratio”.

Numbers like phi are constructible. All constructible numbers are algebraic, i.e. a solution of a polynomial equation with rational number coefficients. Equivalently non algebraic (transcendental) numbers are never constructible. A possible “modern” (non-collapsing compass) construction of \( \phi \) is shown in Figure 1.
Consider now the rectangle ABCD with AE = AD and rectangle FEBC similar to rectangle ABCD.

The similarity condition means that \( \frac{AD}{EB} = \frac{AB}{AD} = \frac{AE+EB}{AD} = \frac{AD+EB}{AD} = 1 + \left( \frac{AD}{EB} \right)^{-1} \). This is precisely the extreme and mean proportion. Hence, \( \phi = \frac{AB}{AD} \) and such a rectangle is called “Golden”. The number \( \phi \) occurs in the solution of many different problems. For example, the famous Fibonacci sequence which is defined by the following recursion,

\[
F_{n+2} = F_{n+1} + F_n \quad ; \quad F_1 = F_2 = 1 \quad ; \quad \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144\ldots \}
\]

has the explicit formula

\[
F_n = \frac{1}{\phi^2 + 1} \left[ \phi^{n+1} + \left( -\frac{1}{\phi} \right)^{n+1} \right] = \frac{1}{\sqrt{5}} \left[ \phi^n - \left( -\frac{1}{\phi} \right)^n \right] \approx \frac{\phi^{n+1}}{\phi^2 + 1} = \frac{\phi^n}{\sqrt{5}}.
\]

Similarly, a “Golden Spiral” can be constructed by inscribing a quarter circle in successive adjacent squares inside a Golden Rectangle. This is shown in Figure 5. The limit point of these spirals can be expressed in terms of \( \phi \). The frequency with which \( \phi \) unexpectedly appears in mathematics has given it an almost magical reputation.
Act II: “The Golden Triangle”

Consider an isosceles triangle with the property that when one of its equal angles is bisected the bisector generates a smaller triangle similar to the original. This is illustrated in Figure 6.

**Figure 6**

The Golden Triangle

\[
\frac{w}{S} = \frac{S-w}{w}
\]

\[
w^2 + Sw - S^2 = 0
\]

\[
\frac{w}{S} = \frac{\sqrt{5} - 1}{2}
\]

\[
S = \frac{2(\sqrt{5} + 1)}{5 - 1} = \phi
\]

Since \(\alpha + 2\alpha + 2\alpha = 180^\circ\), we see that the angles must be \(36^\circ, 72^\circ, 72^\circ\).

From the similarity of the two triangles \(\frac{S}{w} = \frac{w}{S-w} = \frac{1}{\frac{S}{w} - 1}\). This yields the quadratic equation

\[
\left(\frac{S}{w}\right)^2 - \frac{S}{w} - 1 = 0
\]

The only positive solution is \(\frac{S}{w} = \frac{\sqrt{5} + 1}{2} = \phi\).

Bisecting the side opposite to the \(36^\circ\) degree angle gives the result that \(\sin(18^\circ) = \frac{w}{2S} = \frac{\sqrt{5} - 1}{4}\) and

\[
\cos(18^\circ) = \sqrt{1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}
\]

In terms of \(\phi\), \(\sin(18^\circ) = \frac{w}{2S} = \frac{1}{2\phi}\), and

\[
\cos(18^\circ) = \sqrt{1 - \frac{1}{4\phi^2}} = \frac{\sqrt{4\phi^2 - 1}}{2\phi} = \frac{\sqrt{(2\phi - 1)(2\phi + 1)}}{2\phi} = \frac{\sqrt{(2\phi - 1)(\phi^2 + \phi)}}{2\phi} = \frac{\sqrt{(2\phi^2 - \phi)\phi^2}}{2\phi} = \frac{\sqrt{\phi + 2}}{2}
\]
Act III: “Knowing All the Angles”

From the half-angle formulas we have the following “exact results”.

\[
\sin(15^\circ) = \sqrt{\frac{1-\cos(30^\circ)}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \sqrt{\frac{3-2\sqrt{3}+1}{8}} = \sqrt{\frac{\left(\sqrt{3}-1\right)^2}{8}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{6} - \sqrt{2}}{4}
\]

\[
\cos(15^\circ) = \sqrt{\frac{1+\cos(30^\circ)}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \sqrt{\frac{3+2\sqrt{3}+1}{8}} = \sqrt{\frac{\left(\sqrt{3}+1\right)^2}{8}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{6} + \sqrt{2}}{4}
\]

From the “subtraction” formulas,

\[
\sin(3^\circ) = \sin(18^\circ - 15^\circ) = \sin(18^\circ)\cos(15^\circ) - \cos(18^\circ)\sin(15^\circ)
\]

\[
= \left(\frac{\sqrt{5} - 1}{4}\right) \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right) - \left(\frac{\sqrt{10} + 2\sqrt{5}}{4}\right) \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)
\]

\[
= \frac{\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2} - 2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5} + \sqrt{5}}{16}
\]

\[
\cos(3^\circ) = \cos(18^\circ - 15^\circ) = \cos(18^\circ)\cos(15^\circ) + \sin(18^\circ)\sin(15^\circ)
\]

\[
= \left(\frac{\sqrt{10} + 2\sqrt{5}}{4}\right) \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right) + \left(\frac{\sqrt{5} - 1}{4}\right) \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)
\]

\[
= \frac{2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5} + \sqrt{5} + \sqrt{30} - \sqrt{10} - \sqrt{6} + \sqrt{2}}{16}
\]

These results for three degrees can be used to construct exact values for any multiple of three degrees using the following recursions based on the addition formulas.

\[
\sin[(n+1)3^\circ] = \sin(3n^\circ)\cos(3^\circ) + \cos(3n^\circ)\sin(3^\circ)
\]

\[
= \left(\frac{\sqrt{30} - \sqrt{10} - \sqrt{6} + \sqrt{2} + 2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5} + \sqrt{5}}{16}\right)\sin(3n^\circ) +
\]

\[
\left(\frac{\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2} - 2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5} + \sqrt{5}}{16}\right)\cos(3n^\circ)
\]

\[
\cos[(n+1)3^\circ] = \cos(3n^\circ)\cos(3^\circ) - \sin(3n^\circ)\sin(3^\circ)
\]

\[
= \left(\frac{\sqrt{30} - \sqrt{10} - \sqrt{6} + \sqrt{2} + 2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5} + \sqrt{5}}{16}\right)\cos(3n^\circ) -
\]

\[
\left(\frac{\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2} - 2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5} + \sqrt{5}}{16}\right)\sin(3n^\circ)
\]
For some angles additional simplification is possible. For example, from the recursion the cosine of
\[21° = \frac{7\pi}{60}\]
is found to be
\[
\frac{\sqrt{30} + \sqrt{10} + \sqrt{6} + \sqrt{2} + \sqrt{75} + 15\sqrt{5} - \sqrt{25 + 5\sqrt{5}} - \sqrt{15 + 3\sqrt{5}} + \sqrt{5} + \sqrt{5}}{16}.
\]
However,
\[
\sqrt{75 + 15\sqrt{5}} - \sqrt{25 + 5\sqrt{5}} - \sqrt{15 + 3\sqrt{5}} + \sqrt{5} + \sqrt{5} = \\
\sqrt{15 + 3\sqrt{5}(\sqrt{5} - 1)} - \sqrt{5} + \sqrt{5}(\sqrt{5} - 1) = \\
\sqrt{15 + 3\sqrt{5}\sqrt{(\sqrt{5} - 1)^2}} - \sqrt{5} + \sqrt{5}\sqrt{(\sqrt{5} - 1)^2} = \\
\sqrt{(15 + 3\sqrt{5})(6 - 2\sqrt{5})} - \sqrt{(5 + \sqrt{5})(6 - 2\sqrt{5})} = \\
\sqrt{60 - 12\sqrt{5} - \sqrt{20 - 4\sqrt{5}}} = 2\sqrt{15 - 3\sqrt{5} - 2\sqrt{5} - \sqrt{5}}.
\]
So that the cosine of 21 degrees can be expressed as
\[
\cos(21°) = \frac{\sqrt{30} + \sqrt{10} + \sqrt{6} + \sqrt{2} + 2\sqrt{15 - 3\sqrt{5} - 2\sqrt{5} - \sqrt{5}}}{16}.
\]
Proceeding in this manner the results in Table 1 were generated. Angles beyond 45° can be generated by
the symmetry of the unit circle:
\[
\cos(\theta) = \sin(90° - \theta) = \cos(360° - \theta) = -\cos(180° - \theta)
\]
\[
\sin(\theta) = \cos(90° - \theta) = -\sin(360° - \theta) = \sin(180° - \theta)
\]
In fact, multiples of three degrees is the “best” one can do using integer degree arguments. This follows
from Gauss’s famous constructability theorem for regular polygons (3, 4). Regular polygons with 180 and
360 sides can not be constructed with rule and compass since the prime factorization of both of these
numbers contains two factors of the Fermat prime \(3 = 2^2 + 1\). This means that no exact solution in terms
of square roots is possible for the trigonometric functions of either 2° or 1°. Of course 120 can be factored
as 120 = \(2^3 \times 5\) and so contains only one factor each of the Fermat primes 3 and 5. Thus a 120 sided
polygon is constructible and the exact values for three degrees presented above must exist.

You could go further and use the half angle formulas to extend the values in Table 1 to intervals of
0.75° and then interpolate the resulting decimal values. This would allow the construction from exact
values of that peculiar anachronism known as a trig table without the use of an infinite series!
Table 1: Exact Values of Sine and Cosine for Angles that are Multiples of Three Degrees

<table>
<thead>
<tr>
<th>θ</th>
<th>cos θ</th>
<th>sin θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0º = 0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3º = $\frac{\pi}{60}$</td>
<td>$\sqrt{30} - \sqrt{10} - \sqrt{6} + \sqrt{2} + 2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5 + \sqrt{5}}$</td>
<td>$\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2} - 2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5 + \sqrt{5}}$</td>
</tr>
<tr>
<td>6º = $\frac{\pi}{30}$</td>
<td>$\sqrt{15} + \sqrt{3} + \sqrt{10} - 2\sqrt{5}$</td>
<td>$-1 - \sqrt{5} + \sqrt{30} - 6\sqrt{5}$</td>
</tr>
<tr>
<td>9º = $\frac{\pi}{20}$</td>
<td>$\sqrt{10} + \sqrt{2} + 2\sqrt{5} - \sqrt{5}$</td>
<td>$\sqrt{10} + \sqrt{2} - 2\sqrt{5} - \sqrt{5}$</td>
</tr>
<tr>
<td>12º = $\frac{\pi}{15}$</td>
<td>$\sqrt{5} - 1 + \sqrt{30} + 6\sqrt{5}$</td>
<td>$\sqrt{3} - \sqrt{15} + \sqrt{10} + 2\sqrt{5}$</td>
</tr>
<tr>
<td>15º = $\frac{\pi}{12}$</td>
<td>$\sqrt{6} + \sqrt{2}$</td>
<td>$\sqrt{6} - \sqrt{2}$</td>
</tr>
<tr>
<td>18º = $\frac{\pi}{10}$</td>
<td>$\sqrt{10} + 2\sqrt{5}$</td>
<td>$\sqrt{5} - 1$</td>
</tr>
<tr>
<td>21º = $\frac{7\pi}{60}$</td>
<td>$\sqrt{30} + \sqrt{10} + \sqrt{6} + \sqrt{2} + 2\sqrt{15} - 3\sqrt{5} - 2\sqrt{5} - \sqrt{5}$</td>
<td>$-\sqrt{30} + \sqrt{10} - \sqrt{6} + \sqrt{2} + 2\sqrt{15} - 3\sqrt{5} + 2\sqrt{5} - \sqrt{5}$</td>
</tr>
<tr>
<td>24º = $\frac{2\pi}{15}$</td>
<td>$1 + \sqrt{5} + \sqrt{30} - 6\sqrt{5}$</td>
<td>$\sqrt{15} + \sqrt{5} - 10 - 2\sqrt{5}$</td>
</tr>
<tr>
<td>27º = $\frac{3\pi}{20}$</td>
<td>$\sqrt{10} - \sqrt{2} + 2\sqrt{5} + \sqrt{5}$</td>
<td>$-\sqrt{10} + \sqrt{2} + 2\sqrt{5} - \sqrt{5}$</td>
</tr>
<tr>
<td>30º = $\frac{\pi}{6}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>33º = $\frac{11\pi}{60}$</td>
<td>$-\sqrt{30} + \sqrt{10} + \sqrt{6} - \sqrt{2} + 2\sqrt{15} + 3\sqrt{5} + 2\sqrt{5} + \sqrt{5}$</td>
<td>$\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2} + 2\sqrt{15} + 3\sqrt{5} - 2\sqrt{5} + \sqrt{5}$</td>
</tr>
<tr>
<td>36º = $\frac{\pi}{5}$</td>
<td>$\frac{\sqrt{5} + 1}{4}$</td>
<td>$\frac{\sqrt{10} - 2\sqrt{5}}{4}$</td>
</tr>
<tr>
<td>39º = $\frac{13\pi}{60}$</td>
<td>$\sqrt{30} - \sqrt{10} + \sqrt{6} - \sqrt{2} + 2\sqrt{15} - 3\sqrt{5} + 2\sqrt{5} - \sqrt{5}$</td>
<td>$\sqrt{30} + \sqrt{10} + \sqrt{6} + \sqrt{2} - 2\sqrt{15} - 3\sqrt{5} + 2\sqrt{5} - \sqrt{5}$</td>
</tr>
<tr>
<td>42º = $\frac{7\pi}{30}$</td>
<td>$\sqrt{15} - \sqrt{3} + \sqrt{10} + 2\sqrt{5}$</td>
<td>$1 - \sqrt{5} + \sqrt{30} + 6\sqrt{5}$</td>
</tr>
<tr>
<td>45º = $\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
</tbody>
</table>
Act IV: “Constructing a Regular Pentagon on a Limited Budget”

Proposition 10 of Book IV of Euclid’s Elements (5) gives a construction of the Golden Triangle. This serves as the basis for Proposition 11 (6) which constructs a regular pentagon.

However, the number of steps involved is rather large. A very simple and elegant construction of a regular pentagon was given by H. W. Richmond in 1893 (7, 8, 9) and is presented below.

Figure 7

H. W. Richmond's Construction of a Regular Pentagon

(1) Construct a pair of perpendicular lines. Let O mark the point of intersection.
(2) Construct a circle of radius r centered at O.
(3) Mark one of the points of intersection of the lines of (1) with the circle of (2) as A.
(4) On the line which does not contain A bisect the radius at point M.
(5) Bisect the angle $\angle OMA$ and extend the bisector to P on segment OA.
(6) From P construct a line perpendicular to OA with point of intersection C with the circle of (2)
(7) Mark off length AC and proceed around the circle generating three additional vertices.

The following argument based on trigonometry proves the validity of Richmond’s construction.

Since $OA$ is twice $OM$, $\tan(2\angle OMP) = 2$. From the addition formulas,

$$\tan(2\angle OMP) = \frac{2\sin(\angle OMP)\cos(\angle OMP)}{\cos^2(\angle OMP) - \sin^2(\angle OMP)} = \frac{2\tan(\angle OMP)}{1 - \tan^2(\angle OMP)} = 2.$$
Rearranging the above equation gives a quadratic equation:

\[ 1 - \tan^2(\angle OMP) = \tan(\angle OMP) \quad \text{or} \quad \tan^2(\angle OMP) + \tan(\angle OMP) - 1 = 0. \]

The tangent of angle \( OMP \) is positive, so the only admissible solution is

\[ \tan(\angle OMP) = -1 + \frac{\sqrt{5}}{2} = \frac{1}{\phi}. \]

Hence, \( OP = MO \tan(\angle OMP) = \frac{\sqrt{5} - 1}{4} OC \) and \( \sin(\angle OCP) = \frac{OP}{OC} = \frac{\sqrt{5} - 1}{4} = \frac{1}{2\phi} \).

Thus, \( \angle OCP = 18^\circ \); \( \angle COA = 90^\circ - \angle OCP = 72^\circ \), which divides the circle into five equal arcs. The argument is summarized in Figure 8.

**Figure 8**

**H. W. Richmond's Construction of a Regular Pentagon**

**Why It Works**

\[
\begin{align*}
\tan(2\angle OMP) &= \frac{2\sin(\angle OMP)\cos(\angle OMP)}{\cos^2(\angle OMP) - \sin^2(\angle OMP)} - \frac{2\tan(\angle OMP)}{1 - \tan^2(\angle OMP)} - 2 \\
2 - 2\tan^2(\angle OMP) &= 2\tan(\angle OMP) \\
\tan^2(\angle OMP) + \tan(\angle OMP) - 1 &= 0 \\
\tan(\angle OMP) &= -1 + \frac{\sqrt{5}}{2} \\
OP &= MO \tan(\angle OMP) = \frac{\sqrt{5} - 1}{4} OC \\
\sin(\angle OCP) &= \frac{OP}{OC} = \frac{\sqrt{5} - 1}{4} = \frac{1}{2\phi} \\
\angle OCP &= 18^\circ \quad \text{;} \quad \angle COA = 90^\circ - \angle OCP = 72^\circ
\end{align*}
\]
Act V: “Magic Happens”

Line up 5 Golden Triangles in a horizontal row so that their short bases form a continues line segment of width $5w$. Then rotate the four right most triangles $72^\circ$ clockwise. Now rotate the three triangles at the end of the line $72^\circ$ clockwise, next rotate the two triangles at the end of the line $72^\circ$ clockwise, and finally rotate the last triangle $72^\circ$ clockwise. This will generate a regular pentagram. As shown in Figure 9 the ratio properties noted by Beakman and Jax follow rather easily from the Golden Quadratic Equation $\phi^2 = \phi + 1$.

A more complete and revealing analysis considers a regular pentagon inscribed in a circle of radius $OA$ and with vertices at $A, B, C, D$ and $E$. Construct segments $AD, AC, BE, BD$, and $CE$ with points of intersection $F, M, G, H, I$ and $J$. This is shown in Figure 10.
First $\angle AOB = \frac{360^\circ}{5} = 72^\circ$. Then since the measure of an angle inscribed on a circle is equal to half of its subtended arc, we have that $\angle EAD = \angle DAC = \angle CAB = \angle AEB = \angle ABE = \angle BEC = \angle EBD = 36^\circ$. So by ASA congruency AEF and ABG are congruent isosceles triangles with $AF = EF = AG = GB$ so all of the sides of the pentagram $AFEJDICHBG$ are equal. Furthermore, triangle $AFG$ is isosceles so that $\angle AFG = \angle AGF = \frac{180^\circ - \angle DAC}{2} = 72^\circ$, $\angle ADC = \frac{\text{arc} AC}{2} = 72^\circ$, and $\angle AJE = 180^\circ - \angle BEC - \angle AEB - \angle EAD = 180^\circ - 3(36^\circ) = 72^\circ$. Similar arguments establish that $\angle ACD = \angle AHB = 72^\circ$. Thus, as shown in Figure 11 the four triangles $AFG$, $AEJ$, $ABH$ and $ADC$ are “Golden”. In the same way each vertex of the pentagon $ABCDE$ is the “top” vertex of four such “Golden” triangles.
From the triangle $AFG$, $\frac{AF}{FG} = \phi$, but $EF = AF$, so $\frac{EF}{FG} = \phi$. Now $\frac{EG}{EF} = \frac{EF + FG}{EF} = 1 + \frac{1}{\phi}$; however,

$$1 + \frac{1}{\phi} = \frac{\phi + 1}{\phi} = \frac{\phi^2}{\phi} = \phi.$$

Similarly, $\frac{EB}{EG} = \frac{EG + GB}{EG} = \frac{EG + EF}{EG} = 1 + \frac{1}{\phi} = \phi$. So in summary each of the following ratios of lengths of segments in the chord $EB$ is the Golden Ratio:

$$\frac{EF}{FG} = \frac{EG}{EF} = \frac{EG}{GB} = \frac{EB}{EG} = \frac{EB}{FB} = \phi = 1.618033988749... .$$

Triangles $AEJ$ and $ACD$ are both Golden and $FS = FG$, $EJ = EF = AF$, $AJ = AE = CD$. Thus,

$$\frac{AD}{DC} = \frac{AJ}{EJ} = \frac{AF}{EF} = \frac{AJ}{AF} = \frac{AD}{AJ} = \phi.$$

Similarly, $\frac{AH}{AG} = \frac{AC}{GH} = \frac{AH}{BH} = \phi$. 
Finally, exact relationships between the radius, \( r = OA \), of the inscribing circle and segments within the pentagon can be obtained using Table 1.

Since \( \angle AOE = \text{arc}EA = 72^\circ = \angle AOB \) and \( OE = OB = r \) triangles MOB and MOE are congruent.

\[
\angle BMO = \angle EMO = 90^\circ
\]

\[
\frac{MB}{r} = \sin(72^\circ) = \cos(18^\circ) = \frac{\sqrt{10 + 2\sqrt{5}}}{4}
\]

\[
\frac{MO}{r} = \frac{EB}{r} = \frac{2MB}{r} = \frac{\sqrt{10 + 2\sqrt{5}}}{2}
\]

\[
\frac{AM}{r} = 1 - \frac{MO}{r} = \frac{5 - \sqrt{5}}{4}
\]

\[
\frac{AB}{r} = 2\sin(36^\circ) = \frac{\sqrt{10 - 2\sqrt{5}}}{2}
\]

The perimeter, \( P \), of the regular pentagon is related to the radius of the inscribed circle as

\[
\frac{P}{r} = \frac{5\sqrt{10 - 2\sqrt{5}}}{2} = 5.8778525229247... \quad \text{and} \quad \frac{r}{P} = \frac{\sqrt{50 + 10\sqrt{5}}}{50} = 0.17013016167041...
\]

\( FG = GH = HI = IJ = JF \) and \( \angle EFA = \angle EJD = \angle DIC = \angle CHB = \angle BGA = 180^\circ - 2(36^\circ) = 108^\circ \), so \( GFJIH \) is also a regular pentagon. As shown in Figure 12 one can construct an “infinite regress” of imbedded regular pentagons and associated pentagrams. The dimensions of each new pentagon/pentagram are related to the previous pentagon/pentagram by the scale factor \( \frac{OG}{r} \). \( OG \) bisects \( \angle AOB \) so that

\[
\frac{MO}{OG} = \cos(36^\circ) = \frac{\sqrt{5} + 1}{4} \quad \text{and} \quad \frac{OG}{r} = \frac{MO}{r} \left( \frac{\sqrt{5} + 1}{4} \right)^{-1} = \frac{\sqrt{5} - 1}{\sqrt{5} + 1} = \frac{3 - \sqrt{5}}{2} = 0.381966011250105....
\]

The areas of the embedded figures are scaled by

\[
\left( \frac{3 - \sqrt{5}}{2} \right)^2 = \frac{7 - 3\sqrt{5}}{2} = 0.14589803375032... .
\]
References: