Table of Contents

Table of Contents ................................................................................................................................................... 2
Chapter 1: Pre-Algebra ........................................................................................................................................ 3
  Section 1.0: Calculator Use ...................................................................................................................... 3
  Section 1.1: Fractions ............................................................................................................................. 3
  Section 1.2: Decimals ............................................................................................................................. 11
  Section 1.3: Significant Digits ............................................................................................................... 19
  Section 1.4: Signed Numbers ................................................................................................................ 26
  Section 1.5: Exponents .......................................................................................................................... 32
  Section 1.6: Order of Operations ........................................................................................................... 42
  Section 1.7: Evaluating and Simplifying Expressions ......................................................................... 45
Chapter 2: Linear Equations and Inequalities ............................................................................................... 59
  Section 2.1: Solving Linear Equations in One Variable .......................................................................... 59
  Section 2.2: Rearranging Formulas and Solving Literal Equations ......................................................... 64
  Section 2.3: Linear Inequalities in One Variable ..................................................................................... 70
  Section 2.4: Applied Problems ............................................................................................................... 71
  Section 2.5: Percent Problems ............................................................................................................... 75
  Section 2.6: Percent Problems from Finance ........................................................................................ 82
  Section 2.7: Direct and Inverse Variation Problems ............................................................................. 85
Chapter 3: Algebra and the Graph of a Line .................................................................................................... 90
  Section 3.1: Graphing a Linear Equation Using a Table of Values .......................................................... 90
  Section 3.2: Graphing a Linear Equation Using the Slope Intercept Method ....................................... 93
  Section 3.3: Graphing a Linear Equation Using Intercepts .................................................................. 98
  Section 3.4: Graphing a Linear Inequality ............................................................................................. 102
  Section 3.5: Solving a System of Two Linear Equations by Graphing ............................................... 106
  Section 3.6: Solving a System of Two Linear Equations by Algebraic Methods ............................... 113
Chapter 4: Measurement ............................................................................................................................... 123
  Section 4.1: Linear Measurements ....................................................................................................... 123
  Section 4.2: Measuring Area .................................................................................................................. 133
  Section 4.3: Measuring Volume .............................................................................................................. 138
  Section 4.4: Conversion Between Metric and English Units ................................................................. 143
Chapter 5: Geometry ...................................................................................................................................... 150
  Section 5.1: Plane geometry .................................................................................................................. 150
  Section 5.2: Radian Measure and its Applications ............................................................................... 171
  Section 5.3: The Volume and Surface Area of a Solid .......................................................................... 174
Chapter 6 - Trigonometry ............................................................................................................................. 184
  Section 6.1 Sine, Cosine and Tangent ..................................................................................................... 184
  Section 6.2 Solving Right Triangles ...................................................................................................... 189
  Section 6.3 The Law of Sines and the Law of Cosines ......................................................................... 194
  Section 6.4 Solving Oblique Triangles .................................................................................................. 200
Chapter 7 - Statistics ..................................................................................................................................... 208
  Section 7.1 Organizing Data .................................................................................................................. 208
  Section 7.2 Graphing ............................................................................................................................ 218
  Section 7.3 Descriptive Statistics ........................................................................................................ 234
Chapter 1: Pre-Algebra

Section 1.0: Calculator Use
Throughout most of human history computation has been a tedious task that was often postponed or avoided entirely. It is only in the last generation that the use of inexpensive handheld calculators has transformed the ways that people deal with quantitative data. Today the use and understanding of electronic computation is nearly indispensable for anyone engaged in technical work. There are a variety of inexpensive calculators available for student use. Some even have graphing and/or symbolic capabilities. Most newer model calculators such as the Casio models fx-300W, fx-300MS, fx-115MS, and the Texas Instruments models TI-30X IIB, TI-30X IIS, TI-34 II enter calculations in standard “algebraic” format. Older calculators such as the Casio fx-250HC and the Texas Instruments TI-30Xa and TI-36X enter some calculations in a “reverse” format. Both types of calculators are priced under twenty dollars, yet possess enough computational power to handle the problems faced in most everyday applications. The Casio fx-300MS is fairly representative of the “newer” format calculators and the TI-30Xa is typical of “older” format ones. While other calculator models have similar or even better features for performing the required computations, the reader will be responsible for learning their detailed use. Never throw away the user’s manual!

In order to perform a computation, the correct keystrokes must be entered. Although calculators differ in the way keystrokes are entered, this text attempts to provide the reader with a couple of different keystroke options for each example problem in this chapter. The reader should practice the order in which to press the keys on the calculator while reading through the examples. This practice will ensure that the reader knows how to use his/her particular calculator. In order to indicate the sequence of keystrokes the following notation will be used. Digits (0 through 9 plus any decimal point) will be presented in normal typeface. Any additional keystrokes will be enclosed in boxes. For example, to multiply 7 times 8, the command sequence will be written as $7 \times 8 = 56$ appears on the display.

Section 1.1: Fractions
Fractions are ratios of whole numbers, which allow us to express numbers which are between the whole numbers. For example,

$2 \frac{2}{3} = 2 + \frac{2}{3}$ is between 2 and 3.

Fractions represent “part of a whole”. Imagine that we have a freight car with eight equal sized compartments. If three of these compartments are full of grain, we would indicate that we have three eighths of a freight car’s worth of grain. This is illustrated below.

Consider a car with eight compartments of which two are full. The fraction of a full car is two eighths. If we look at the same car split into four equal compartments, this same amount of grain fills one fourth of the car.

We arrive at the following result.
We say that such equal fractions while they “look different” are equivalent. To generate equivalent fractions, we can multiply or divide both numerator (the top number) and denominator (the bottom number) by a common number. So we have the following fractions equivalent to two thirds.

\[
\frac{2}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9} = \frac{2 \times 17}{3 \times 17} = \frac{34}{51}
\]

Similarly, \( \frac{18}{24} = \frac{6 \times 3}{6 \times 4} = \frac{3}{4} \). This same result could be stated in terms of “canceling” the common factor of 6 between the numerator and denominator.

\[
\frac{1}{2} \times \frac{8}{4} = \frac{6 \times 3}{6 \times 4} = \frac{3}{4}
\]

If a fraction has no common factors between its numerator and denominator, the fraction is in lowest terms.

There are three types of fractions.

1. Proper fractions with the numerator less than (symbolized by <) the denominator. All proper fractions are less than 1.

2. Improper fractions with the numerator greater than (symbolized by >) the denominator. All improper fractions are greater than 1. Improper fractions can be expressed as a mixed number, which is a whole number plus a proper fraction. For example,

\[
\frac{25}{6} = \frac{6 \times 4 + 1}{6} = 4 \frac{1}{6}
\]

3. Unit fractions with the numerator equal to the denominator. All unit fractions are equal to one. For example,

\[
\frac{19}{4} = \frac{25}{1}
\]
Note: when we write $4\frac{1}{6}$, we are using a shorthand notation. There really is a $+$ sign between the 4 and the one sixth that’s understood but unstated. Working backwards we can convert a mixed number into an improper fraction. For example,

$$\frac{7\frac{2}{3}}{3} = \frac{7 \times 3 + 2}{3} = \frac{23}{3}.$$

Fractions can be entered on many calculators using the $\frac{a}{b}$ key. For example, use the following keystrokes to enter the fraction $\frac{14}{24}$: $14,a,b,c \div 24$, $=$. $\frac{7}{12}$ will then appear in the display as the fraction reduced to lowest terms.

For a mixed number such as, $11\frac{5}{6}$ enter the following keystrokes:

$11,a,b,c \div 5,a,b,c \div 16$, $=$. Some calculators display $11\frac{5}{6}$, while other calculators display $11\frac{5}{6}$. To change this answer to the improper fraction $\frac{181}{16}$, enter $\text{shift} a,b,c$ on some calculators, or $\text{2nd} a,b,c$ on other calculators. Also, on some calculators the $a,b,c$ key when pressed after entering a fraction converts it to a decimal and if $a,b,c$ is pressed a second time the decimal is converted back to a fraction. On other calculators fraction – decimal conversions are performed by entering $\text{2nd} \leftarrow$. The $\leftarrow$ key is also the back space key, which deletes characters in the display.

As an application, solve for the following missing numerator: $\frac{6}{8}$. As a first step reduce $\frac{6}{8}$ to lowest terms as $\frac{3}{4}$. So $\frac{3}{12} = \frac{3}{4}$. The first denominator 12 is three times the second denominator 4, so the missing numerator must be three times the second numerator 3. The answer is that the missing numerator is 9.

To compare two fractions and determine which is larger, we can use the following procedure:

1. If the fractions involve mixed numbers with proper fractions, the number with the larger whole number is the larger number. For example,

$$7\frac{3}{16} > 5\frac{7}{8}, \text{ since } 7 > 5.$$

2. If the fractions are both proper fractions or mixed numbers with equal whole numbers, then convert the fractions into decimals. The number with the larger decimal is the larger number. For example,
\[
5 \frac{1}{3} < 5 \frac{3}{8}, \text{ since } \frac{3}{8} = 0.375 > 0.333... = \frac{1}{3}.
\]

To add or subtract fractions we need a common denominator. Consider adding \(\frac{3}{8}\) to \(\frac{1}{4}\). Since one fourth is equivalent to two eighths, we have the following solution:

\[
\begin{align*}
\frac{3}{8} + \frac{1}{4} &= \frac{3}{8} + \frac{2}{8} \\
&= \frac{5}{8}
\end{align*}
\]

If mixed numbers are involved, we first deal with the whole numbers, then the fractions. For example,

\[
7 \frac{1}{2} - 5 \frac{9}{16} = 7 - 5 + \frac{1}{2} - \frac{9}{16} = 2 + \frac{8}{16} - \frac{9}{16} = 1 + \frac{16}{16} + \frac{8}{16} - \frac{9}{16} = 1 + \frac{16+8-9}{16} = 1 + \frac{15}{16} = \frac{15}{16}.
\]

Note: Since \(\frac{9}{16} > \frac{8}{16}\), we had to “borrow” \(\frac{16}{16}\) from the 2.

To multiply fractions we form the product of the numerators over the product of the denominators. For example,

\[
\frac{5}{8} \times \frac{3}{4} = \frac{5 \times 3}{8 \times 4} = \frac{15}{32}.
\]

If the product involves mixed numbers, we first convert them to improper fractions. For example,

\[
2 \frac{5}{6} \times 4 \frac{1}{5} = \frac{2 \times 6 + 5}{6} \times \frac{4 \times 5 + 1}{5} = \frac{17}{6} \times \frac{21}{5} = \frac{17 \times 7}{3 \times 2} \times \frac{3 \times 7}{5} = \frac{17 \times 7}{2 \times 5} = \frac{119}{10} = 11 \frac{9}{10}.
\]

Note: we canceled the common factor of 3 between numerator and denominator in this calculation. In a multiplication problem this can always be done and saves the effort of later having to reduce the final answer. Also note that the answer is “reasonable” in that

\[
2 \frac{5}{6} \approx 3 \text{ and } 4 \frac{1}{5} \approx 4, \text{ so } 2 \frac{5}{6} \times 4 \frac{1}{5} \approx 3 \times 4 = 12.
\]

A quick estimation like this can often catch silly mistakes even when using a calculator!

Consider the division problem \(8 \div 2 = 4\). This is the same as

\[
\frac{8}{2} = \frac{8 \times \frac{1}{2}}{2} = \frac{8 \times 1}{2} = 4.
\]

More generally, any division problem can be expressed as
This means that division by the number \( b \) is equivalent to multiplication by the fraction \( \frac{1}{b} \). The fraction \( \frac{1}{b} \) is called the \textbf{reciprocal} of \( \frac{b}{1} \). To form the reciprocal of a number we exchange the numerator with the denominator. In summary, division by a \textbf{non-zero} number equals multiplication by the reciprocal of that number.

In a division problem \( 0 \) is never allowed as the denominator or divisor. The reason for this is as follows.

Suppose \( 20 \div 0 = \frac{20}{0} \) made sense. Then there would be some number, \( a \), which is the answer to this division problem. Restating this as a multiplication problem would give \( a \times 0 = 20 \). But any number times zero gives zero! So no sensible answer to \( 20 \div 0 = \frac{20}{0} \) exists.

Another way of explaining this goes to the very meaning of division. \( 20 \div 4 = \frac{20}{4} = 5 \), means that 20 contains five 4’s. How many 0’s does 20 contain? There’s no sensible answer to the question!

Consider now the division \( \frac{1}{4} \div \frac{1}{8} \), from the diagram below it is clear that one fourth contains 2 one eighths. So the answer must be 2. The following shows that this result is consistent with the multiplication by the reciprocal definition of division.

\[
\frac{1}{4} \div \frac{1}{8} = \frac{1}{4} \times \frac{8}{1} = \frac{8}{4} = 2.
\]

If the division involves mixed numbers, we first convert them into improper fractions. For example,

\[
4 \frac{1}{2} \div 1 \frac{1}{8} = \frac{9}{2} \div \frac{9}{8} = \frac{9}{2} \times \frac{8}{9} = \frac{8}{2} = 4.
\]
In expressions which combine operations the standard order of operations apply as shown in the following:

\[
\frac{2}{3} \times \frac{1}{2} - \frac{1}{4} \div \frac{3}{2} - \frac{5}{4} \times \frac{1}{3} = \frac{5}{3} - \frac{5}{12} = \frac{5 \times 4 \times 15}{3 \times 4} = \frac{1}{3} = \frac{3 \times 1}{3 \times 4} = \frac{1}{4}.
\]

These calculations are all easily performed on the calculator. The keystrokes for the previous calculation are as follows:

\[
2 \div \frac{1}{3} \times \frac{2}{1} - \frac{3}{2} - \frac{1}{15} = \frac{4}{3} \times \frac{12}{12} \times \frac{12}{12} = \frac{1}{3} \times \frac{1}{4}.
\]

More involved calculations with grouping symbols are also possible. For example,

\[
\frac{3}{4} - \left( \frac{5}{3} - \frac{7}{8} \right) \div \frac{1}{2} = \frac{3}{4} - \left( \frac{2 + \frac{3}{16} - \frac{14}{16}}{\frac{5}{2}} \right) = \frac{3}{4} - \left( \frac{1 + \frac{16 + 3 - 14}{16}}{\frac{5}{2}} \right) = \frac{3}{4} \times \frac{1}{5} = \frac{3}{4} \times \frac{1}{5} = \frac{3}{21} = \frac{3}{9} = \frac{30 - 21}{40}.
\]

This is keystroked as follows:

\[
3 \div \frac{5}{3} \times \frac{16}{3} \times \frac{7}{8} = \frac{2}{1} - \frac{3}{2} - \frac{1}{2} = \frac{4}{3} \times \frac{12}{12} \times \frac{12}{12} = \frac{1}{3} \times \frac{1}{4}.
\]

**Your Turn!!**

Write as an improper fraction.

1) \[ \frac{6}{16} \]

2) \[ \frac{3}{10} \]

Write as a mixed number reduced to lowest terms.

3) \[ \frac{19}{8} \]

4) \[ \frac{26}{10} \]
Reduce to lowest terms.

\[
\frac{9}{12} \quad 5)
\]

\[
\frac{7}{32} \quad 6)
\]

Supply the missing numerators:

\[
\frac{3}{4} = \frac{?}{12} \\
7)
\]

\[
\frac{3}{4} = \frac{?}{8} \\
8)
\]

Indicate which number is larger.

\[
\frac{19}{32} \quad \frac{5}{8} \quad 9)
\]

\[
\frac{3}{16} \quad \frac{7}{4} \quad 10)
\]

Perform the indicated operations and express the answer as a fraction in lowest terms:

\[
\frac{3}{16} \times \frac{4}{9} \quad 11)
\]

\[
6 \div \frac{2}{3} \quad 12)
\]

\[
\frac{3}{8} \div \frac{5}{8} \quad 13)
\]

\[
\frac{13}{32} \quad \frac{9}{32} \quad 14)
\]
Solve and state all results as fractions reduced to lowest terms.

How many pieces of $\frac{5}{16}$ inch thick plywood are in a stack 35 inches high?

23) ________________

A lumberyard sells lumber only in even foot lengths. What is the shortest single board of lumber from which a carpenter could cut three $\frac{3}{4}$ ft long and two $\frac{3}{4}$ ft long pieces?

24) ________________

A cubic foot contains about $\frac{7}{2}$ gallons. How many cubic feet are there in 120 gallons?

25) ________________

A nail $3\frac{1}{2}$ inches long, goes through a board $2\frac{3}{8}$ inches thick supporting a joist. How far into the joist does the nail extend?
26) ________________

A part is measured as 2\(\frac{7}{16}\) inches long on a scale drawing. If the scale is one foot to \(\frac{1}{2}\) inch, how long is the actual part?

27) ________________

**Section 1.2: Decimals**

The difficulty in adding or subtracting fractions “by hand” compared to adding or subtracting whole numbers is obvious to anyone who has done such calculations. This difficulty motivated the development of representing fractions as decimal numbers. Using decimal fractions all arithmetical operations are similar to computations with whole numbers. The only complication is keeping track of the position of the decimal point.

The basis of the decimal representation of numbers is the use of **place value**. This allows us to represent an infinite range of numbers with only ten symbols (the digits 0 through 9). Contrast this with Roman Numerals or other early number systems where new symbols are constantly added to represent larger values. Place value uses the powers of 10.

\[
egin{align*}
10^0 &= 1 \quad \text{(The definition of an exponent of 0)} \\
10^1 &= 10 \\
10^2 &= 100 \\
10^3 &= 1,000 \\
10^4 &= 10,000 \\
10^5 &= 100,000 \\
10^6 &= 1,000,000 \\
\text{etc.}
\end{align*}
\]

**Note:** \(10^n\) is equal to 1 followed by \(n\) zeros.

When we write a number such as 27,483, the digit 2 stands not for 2, but for \(2(10^4) = 20,000\). The digit 7 represents \(7(10^3) = 7000\), etc. The value we associate with each digit comes from its place in the number. The right most digit of a whole number is in the “one’s place”, the second digit from the right is the “ten’s place”, etc. To extend the decimal system to fractions, we use the reciprocal powers of 10 and the decimal point to separate the “one’s place” from the “tenth’s place”. 

The leading 0 to left of the decimal point is not required for a number smaller than 1. It is used to emphasize the location of the decimal point. A decimal fraction such as 0.375 is interpreted as

\[
0.375 = 3 \times 10^{-1} + 7 \times 10^{-2} + 5 \times 10^{-3} \\
= \frac{3}{10} + \frac{7}{100} + \frac{5}{1000} \\
= \frac{375}{1000} = \frac{3 \times 125}{8 \times 125} = \frac{3}{8}.
\]

**Note:** adding extra zeros to the right of the rightmost digit to the right of the decimal point does **not** change the value of the decimal fraction. It does, however, imply a greater knowledge of the precision of the value.

A decimal fraction like 0.375 is called a **terminating** decimal because the digits to the right of the decimal point come to an end. The procedure outlined above is how to convert a terminating decimal to a fraction. It is summarized below:

1. Carry along the digits to the left of the decimal point as the whole number part of the resulting mixed number. If there are no non-zero digits to the left of the decimal point, the decimal represents a proper fraction.

2. Put the digits to the right of the decimal point over the power of 10 that goes with the rightmost decimal place. For example, in converting 0.1145, 1145 is put over 10,000 since the right most digit, 5, is in the ten-thousandth’s place.

\[
0.1145 = \frac{1145}{10000} = \frac{5 \times 229}{5 \times 2000} = \frac{229}{2000}.
\]

3. Reduce this fraction to lowest terms.

To convert a fraction to a decimal is quite easy. We just translate the fraction bar into a division. Remember that in a mixed number there is an understood but unstated plus sign. So that

\[
7 \frac{11}{16} = 7 + \frac{11}{16} = 7.6875.
\]

This can also be done directly using the calculator as was discussed in **Section 1.3 - Fractions.**
If a fraction is in lowest terms and its denominator has a factor besides 2 or 5, then that fraction, when converted to a decimal, will generate a repeating decimal. For example,

\[
\frac{5}{12} = \frac{5}{2 \times 2 \times 3}, \text{ so 12 has a factor of 3, and } 5 \div 12 = 0.416666... = 0.4\overline{16}. 
\]

The 6’s as indicated either by the ellipsis “…” or 6 with a bar on top repeat “forever”.

**Note:** all of these ways of writing the repeating decimal are the same. Calculators will display 0.41666667 since they work with a fixed number of digits and will round the last digit displayed.

To convert a repeating decimal into a fraction is a little complicated and is rarely encountered in practical problems. As a result no problems requiring such a conversion occur in the unit exercises. However, if you are curious, the procedure is summarized and illustrated below:

1. Count and record the number of decimal places from the decimal point to the repeating string of digits.

2. Move the decimal point to the right by this number of places. The result is a decimal number where the repeating pattern of digits begins in the tenth’s place immediately to the right of the decimal point.

3. The digits to the left of the decimal point of the result from Step 2 become the whole number part of a mixed number. If there are no non-zero digits to the left of the decimal point, then the original decimal began the repeating pattern with the first digit and the whole number part of the mixed number is zero.

4. Add the whole number from Step 3 to a fraction with the repeating digits as the numerator and a string of 9’s as the denominator. The number of 9’s in the string is equal to the number of repeating digits in the numerator.

5. Take the fraction from Step 4 and divide it by 10 raised to the power of the number from Step 1. This number, worked out as a fraction, is the fraction equivalent to the original repeating decimal.

To illustrate the steps convert 0.00666… to a fraction.

Step 1. The number of places from the decimal point to the repeating string of 6’s is two.

Step 2. The result is the decimal 0.666….

Step 3. The whole number is 0.

Step 4. There is one repeating digit, a 6, so the result is \(0 + \frac{6}{9} = \frac{2}{3}\).

Step 5. Dividing two thirds by \(10^2 = 100\) gives

\[
\frac{\frac{2}{3}}{10^2} = \frac{2}{3} \div 100 = \frac{2}{3} \times \frac{1}{100} = \frac{2 \times 1}{3 \times 50 \times 3} = \frac{1}{150}. \text{ So } 0.006 = \frac{1}{150}. 
\]
As a more complicated example consider converting $3.1527272727\ldots$ to a fraction.

Step 1. The number of places from the decimal point to the repeating string of 27’s is two.

Step 2. The result is the decimal $315.272727\ldots$.

Step 3. The whole number is $315$.

Step 4. There are two repeating digits, 27, so the result is $315 + \frac{27}{99} = 315 + \frac{9 \times 3}{9 \times 11} = 315 \frac{3}{11}$.

Step 5. Dividing the answer of Step 4 by $10^2 = 100$ gives

$$315 \frac{3}{11} \div 100 = \frac{3468}{11} \times \frac{1}{100} = \frac{4 \times 867}{11 \times 4 \times 25} = \frac{867}{275} = 3 \frac{42}{275}.$$

Using a calculator we can verify that $\frac{42}{275} = 3 + 42 \div 275 = 3.15272727\ldots = 3.15\overline{27}.$

Often we wish to approximate a decimal number by finding another decimal roughly equal to the first number, but expressed with less digits. This process is called rounding. To round use the following procedure:

1. Determine the decimal place to which the number is to be rounded. Often this is stated in the problem or application.

2. If the digit to the right of this decimal place is less than 5, then replace all digits to the right of this decimal place by zeros or discard them if they are to the right of the decimal point.

3. If the digit to the right of the decimal place is 5 or greater, then increase the digit in this decimal place by 1 and replace all digits to the right of this decimal place by zeros or discard them if they are to the right of the decimal point.

As an example, consider rounding 10,547.395 to the different decimal places shown in the following table.

<table>
<thead>
<tr>
<th>10,547.395 rounded to</th>
<th>Decimal Place of Rounding</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 places</td>
<td>hundredth’s place</td>
<td>10,547.40</td>
</tr>
<tr>
<td>1 place</td>
<td>tenth’s place</td>
<td>10,547.4</td>
</tr>
<tr>
<td>the nearest unit</td>
<td>one’s place</td>
<td>10,547</td>
</tr>
<tr>
<td>the nearest ten</td>
<td>ten’s place</td>
<td>10,550</td>
</tr>
<tr>
<td>the nearest hundred</td>
<td>hundred’s place</td>
<td>10,500</td>
</tr>
<tr>
<td>the nearest thousand</td>
<td>thousand’s place</td>
<td>11,000</td>
</tr>
</tbody>
</table>

Raising numbers to powers or exponents occurs in many applications. Recall that $b^n$ means a product of $n$ factors of $b$. The number $b$ is called the base, and $n$ is the power or exponent.

So $1.574^5 = 1.574 \times 1.574 \times 1.574 \times 1.574 \times 1.574 = 9.661034658$.

This result is correct to as many places as your calculator will display. To perform this calculation on some calculators use the keystrokes $1.574 \left[ ^x \right] 5 \left[ = \right], \text{while other calculators enter } 1.574 \left[ \text{x}^y \right] 5 \left[ = \right].$

Newer and/or graphing calculators generally use the “carrot” symbol $^x$ for exponents.
Exponents of two and three are very common and have special names; $b^2$ is called “b squared” and $b^3$ is called “b cubed”. Many calculators have $x^2$ keys to square a number. When evaluating an expression, the standard order of operations requires that bases be raised to powers before any multiplications or divisions are performed. This hierarchy is built into scientific calculators.

For example, consider evaluating $3.54 \times 7.21^3 - (10.7 \times 6.28)^2 \div 3.56$.

On some calculators this is done with the following keystrokes:

```
3.54 x 7.21 y^ 3 - ( 10.7 x 6.28 ) x^2 \div 3.56 =
```

The display shows the answer as $58.46760266$. The keystrokes on other calculators are identical except that the $y^x$ key is used instead of the $y^x$ key. Some calculators have an $x^3$ key, and this key could have been used instead of $y^x 3$ above.

Consider evaluating $25^{12}$. Entering $25 y^x 12 = $ on the some calculators gives the display $5.960464477 \times 10^{16}$. Entering $25 y^x 12 = $ on other calculators results in $5.960464478 \times 10^{16}$.

Because of the large size of the number both calculators have expressed the result in scientific notation. In scientific notation we express the answer as a decimal number between 1 and 10 times ten to a power. Here the number between 1 and 10 is 5.960464478 and the power on 10 is 16. In ordinary decimal notation, which the calculator can’t display for lack of space, this answer would be written as 59,604,644,780,000,000. If you try to work with these large decimal numbers, the advantages of scientific notation soon become obvious!

**Note:** both calculators seem to suggest that the exponent applies to $5.960464478$. This is not true. The exponent is on ten, but to save space in the display the calculator does not show the 10.

Now consider $(0.04)^{12}$. Most calculators display $1.6777216 \times 10^{-17}$. The result is in scientific notation with a negative exponent on 10. In ordinary decimal notation this result would be $0.000000000000016777216$. The left-most non-zero digit, 1, is 16 ($17–1$) decimal places to the right of the decimal point. Thus, in scientific notation a positive exponent on 10 gives the number of decimal places the decimal point must move to the right to get the ordinary decimal answer, while a negative exponent on 10 gives the number of decimal places the decimal point must move to the left to get the ordinary decimal answer.

To enter a number in scientific notation on some calculators, use the [EXP] key. For example, to enter $6 \times 10^3$ use the following keystrokes: $6.02 [\text{EXP}] 23$. A very small number like $7.15 \times 10^{-12}$ is entered with $7.15 [\text{EXP}] (-) 12$. Here $(-)$ is the “change sign” or minus key. The procedure used on some other calculators is identical except that the [EE] key is used instead of the [EXP] key and the change sign key is $+ \leftrightarrow -$. Scientific notation will be covered more thoroughly in Section 2.5.
Consider a table of squares of the whole numbers.

<table>
<thead>
<tr>
<th>N</th>
<th>N^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
</tr>
<tr>
<td>7</td>
<td>49</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
</tr>
<tr>
<td>9</td>
<td>81</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>11</td>
<td>121</td>
</tr>
<tr>
<td>12</td>
<td>144</td>
</tr>
</tbody>
</table>

If we reverse this table, i.e., start with N^2 and get the value of N, the table would look like.

<table>
<thead>
<tr>
<th>N^2</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1.41421356</td>
</tr>
<tr>
<td>3</td>
<td>1.732050808</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>2.236067977</td>
</tr>
<tr>
<td>6</td>
<td>2.449489743</td>
</tr>
<tr>
<td>7</td>
<td>2.645751311</td>
</tr>
<tr>
<td>8</td>
<td>2.828427125</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>3.16227766</td>
</tr>
<tr>
<td>11</td>
<td>3.31662479</td>
</tr>
<tr>
<td>12</td>
<td>3.464101615</td>
</tr>
</tbody>
</table>

The second number is called the square root of the first. In symbols $N = \sqrt{N^2}$, for example, $3 = \sqrt{9}$. Remember that the square root symbol acts as a grouping symbol. Any operations inside the square root need to be completed before the root is taken. For example, $\sqrt{16 - 16} = \sqrt{100} = 10$.

To perform this computation on the calculator parentheses need to be inserted around the expression inside the square root symbol. On the some calculators enter $\sqrt{16 - 16} = \sqrt{100} = 10$, while $\sqrt{16 - 16}$ are the corresponding keystrokes on other calculators.
A table of the cubes of whole numbers can also be formed.

<table>
<thead>
<tr>
<th>N</th>
<th>N^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>64</td>
</tr>
<tr>
<td>5</td>
<td>125</td>
</tr>
</tbody>
</table>

If we reverse this table, i.e., start with N^3 and get the value of N, the table would look like.

<table>
<thead>
<tr>
<th>N^3</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1.25992105</td>
</tr>
<tr>
<td>3</td>
<td>1.44224957</td>
</tr>
<tr>
<td>4</td>
<td>1.587401052</td>
</tr>
<tr>
<td>5</td>
<td>1.709975947</td>
</tr>
<tr>
<td>6</td>
<td>1.817120593</td>
</tr>
<tr>
<td>7</td>
<td>1.912931183</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

The second number is called the cube root of the first. In symbols, \( N = \sqrt[3]{N^3} \), for example, \( 3 = \sqrt[3]{27} \). The cube root, like the square root, acts as a grouping symbol. Any operations inside the cube root need to be completed before the root is taken. For example,

\[
\sqrt[3]{85 \times 2 - 45} = \sqrt[3]{70 - 45} = \sqrt[3]{25} = 5
\]

To perform this computation on the calculator parentheses need to be inserted around the expression inside the cube root symbol. On some calculators you will enter

SHIFT \( \sqrt[3]{(85 \times 2) - 45} \), while the keystrokes on other calculators

are \( (85 \times 2) - 45 \) \( 2nd \) 0.

**Your Turn!!**

Perform the indicated operations giving answers to the stated number of decimal places:

1) \( 7.11643 + 3.3489 \) (four places)

2) \( 27.32 - 6.972 \) (two places)
0.25 \times 0.4333 \quad \text{(two places)} \quad 3) \quad \underline{\phantom{.00}}

7.123 \div 1.48 \quad \text{(three places)} \quad 4) \quad \underline{\phantom{.00}}

$57.23 - 7.89 \quad \text{(two places, i.e., nearest penny)} \quad 5) \quad \underline{\phantom{.00}}

1.79^2 \quad \text{(two places)} \quad 6) \quad \underline{\phantom{.00}}

Write the following fractions as decimals:

\[
\frac{2\frac{11}{16}}{} \quad 9) \quad \underline{\phantom{.00}}
\]

\[
\frac{\frac{7}{12}}{} \quad 10) \quad \underline{\phantom{.00}}
\]

Write the following decimals as a fraction in lowest terms:

0.45 \quad 11) \quad \underline{\phantom{.00}}

8.84 \quad 12) \quad \underline{\phantom{.00}}

Solve the following problems:

A stack of eighteen pieces of lumber is 31.50 inches thick. How thick would a stack of thirty-three such sheets be?

13) \quad \underline{\phantom{.00}}
A delivery truck gets 11.3 miles per gallon of gasoline. If gas costs $4.12 per gallon, what will be the cost of the gasoline needed to drive 189 miles? (Round to the nearest penny.)

14) ________________

A machinist earns $12.50 an hour plus time and a half for overtime (hours worked beyond 40). What is the machinist’s gross pay for a 53.75 hour work week? (Round to the nearest penny.)

15) ________________

A welder earned $468.75 (gross pay before deductions) for 37.5 hours of work. Find her hourly rate of pay.

17) ________________

A cubic foot holds 7.481 gallons. A car has a gas tank which holds 14.5 gallons. To three decimal places, how many cubic feet is this?

18) ________________

Section 1.3: Significant Digits

In the previous section, each problem told you how to round your answers. In this section, you will learn how to determine for yourself how to round answers when you are working with calculations involving measurements. This will be possible because the way we write a number from a measurement carries information about the precision of the measurement. That, in turn, tells us how to round answers from calculations that involve measurements.
By the end of this section, you should be able to identify the number of significant digits in a measurement number, write the result of a measurement with the correct number of significant digits, and round calculations involving measurements to the correct number of significant digits.

Section 1.3.1: Identifying and Writing Significant Digits

- There is a shortcut that scientists and engineers use to state the precision of a measurement when writing down the measurement.
- For example, if you measure the width of a box of chalk with a standard ruler, you might get an answer of 6.1 cm. Since the ruler is only marked in increments of 0.1 cm, you really can’t measure any finer than half that increment, which is \( \pm 0.05 \) cm. Thus, the best way to state your measurement is as 6.1 cm \( \pm 0.05 \) cm. That way, people know exactly how precise your measurement is, and they know not to expect more precision (like 6.139 cm), and that a stated answer of 6 cm is neglecting the full precision of the measurement. Note: 6.1 cm \( \pm 0.05 \) cm = 61 mm \( \pm 0.5 \) mm.
- If you measure the width with a high-precision instrument like a digital caliper, then you might discover the answer to be 61.22 mm. Again, the uncertainty would be \( \pm \) one-half of the smallest measurement possible, which would be \( \pm 0.005 \) mm. Thus, we’d write the measurement as 61.22 mm \( \pm 0.005 \) mm.
- However, people have found a more efficient way to write a measurement like 6.1 cm \( \pm 0.05 \) cm. Instead, they just write 6.1 cm, and it is understood that the precision is \( \pm \) one-half of the farthest-right digit. In this case, significant digits are being used to indicate the precision of the measurement.
- The significant digits (also called significant figures and abbreviated sig figs) of a number are those digits that carry meaning contributing to its precision.

Example: Suppose that someone tells you the width of a piece of paper is 21.59 cm. However, you know that they measured it with a ruler only marked in increments of 0.1 cm. Write the measurement correctly using significant digits.
Answer: The measurement can only be stated to the nearest 0.1 cm. Thus, it should be written as 21.6 cm.

Example: Suppose you measure the length of a board to be \( 13\frac{5}{8} \) inches, and your tape measure is marked in eighths of an inch. Correctly write the measurement as a decimal using significant digits.
Answer: If you go straight to your calculator, you get \( (13 + 5 \div 8) \text{in.} = 13.625 \text{in.} \). The uncertainty in the measurement is \( \pm \frac{1}{16} \text{in.} \approx 0.0625 \text{in.} \). Thus, the answer is 13.625 in. \( \pm 0.0625 \text{in.} \), which we would write with significant figures as 13.6 in. Or, in other words, \( (1/8)\text{in.} = 0.125 \text{ in.} \), so you should round to the nearest tenth.

Example: Suppose you measured the length of a section of a running track for a sprint race to be exactly 100 meters using a tape that measures to the nearest centimeter (i.e., hundredth of a meter). Write the measurement correctly using significant digits.
Answer: 100.00 m; You have to have all those zeros to show that after the hundreds place, you measured nothing but zeros down to the nearest 0.01 m.

Example: What if you made the same measurement, but the tape only measures to the nearest decimeter (i.e., tenth of a meter)?
Answer: 100.0 m; There is one less zero in the answer to show you only measured to the nearest 0.1 m.

Example: What if the tape only measures to the nearest meter?
Answer: 100. m; The decimal point after the ones place indicates you only measured to the nearest 1 m.

Example: What if the finest increment of the tape is ten meters?
Answer: Either $100m$ or $1.0 \times 10^2$ m; The bar over the tens zero indicates you only measured to the nearest 10 m, while the use of scientific notation (coming up later) shows exactly two significant digits.

Example: What if the finest increment of the tape is 100 meters?
Answer: 100 m;

- These track length examples embody one of the most confusing parts of working with significant digits: measurements that have a lot of zeros in them.
  - When you want to show that a zero is significant, you go to the extra effort of writing extra stuff that you normally wouldn’t write, like putting extra zeros after the decimal point to show the finest increment of measurement:
    - Example: 100.00 m
  - Continuing the theme of writing something a little extra to show what zeros are significant, an old-fashioned trick for showing which trailing zeros are significant is to put a bar over every significant zero:
    - Like $100m$ or $\overline{100}m$ for the last two examples above.
  - A more widely used notation to show when a measurement has been made to the nearest ones place is to put a decimal after the ones place:
    - $100 \text{ m} \iff 100. \text{ m}$

- The rule for identifying the significant digits in a measurement can be stated in two ways:
  - All digits are significant except leading zeros after a decimal point and trailing zeros before it. Both types of zeros merely serve as placeholders.
  - “point right, otherwise left”:
    - If there is a decimal point present, find the left-most nonzero digit, and then count digits toward the right. If there is no decimal point in the number, find the right-most nonzero digit and count toward the left. In both cases, keep counting digits until you reach the other end of the number.

Examples: State the number of significant digits in each measurement below

1. 467.24 ft.
   - Has 5 significant digits

2. 0.006 020 m
   - Has 4 significant digits; the leading zeros are place holders, but the trailing zero is not; it shows you made a measurement down to the nearest 0.000 001 m.

3. 1250 lb.
   - Has 3 significant digits; the trailing zero is a place holder. The measurement is thus only precise to the nearest 10 pounds. If the measurement were accurate to the nearest pound, we would write it as 1250. lb.
4. 0.00003 g  
   • Has 1 significant digit  

5. 93,000,000 mi  
   • Has 2 significant digits  

6. 800 in.  
   • Has 1 significant digit  

7. 800. in.  
   • Has 3 significant digits  

**Section 1.3.2: Significant Digits after Adding or Subtracting**  
- When you add or subtract measurements, then you have to round your answer to the precision of the least accurate measurement. That’s because the uncertainty in the least precise measurement overwhelms the precision of the other measurements.  
- A good practice as you are learning how to work with significant figures is to write down what your calculator gives for an answer, then write how you should round the answer, then write the correctly rounded answer.

\[
\begin{array}{c}
28.25 \text{ cm} \\
+15.67 \text{ cm}
\end{array}
\]
\[
\begin{array}{c}
114.37 \text{ cm} \\
3.080 \text{ cm}
\end{array}
\]
\[
\begin{array}{c}
+27.3 \text{ cm}
\end{array}
\]

**Examples:**
1. \[22.85 \text{ g} - 13.35 \text{ g} = 9.5 \text{ g} \Rightarrow \text{answer must be rounded to a precision of 0.01 g} \Rightarrow = 9.50 \text{ g}\]
2. \[13 \text{ ft} - 5.811 \text{ ft} = 7.189 \text{ ft} \Rightarrow \text{answer must be rounded to a precision of 1 ft} \Rightarrow = 7 \text{ ft}\]
3. \[1250.27 \text{ mi} + 3367.7 \text{ mi} + 2257 \text{ mi} + 4800 \text{ mi} = 11,944.97 \text{ mi} \Rightarrow \text{answer must be rounded to a precision of 100 mi} \Rightarrow = 11,900 \text{ mi}\]

**Section 1.3.3: Significant Digits after Multiplying or Dividing**  
- Rule: Find the measurement with the fewest significant figures. Round your answer to that many significant figures.  
- The rule has to do with how the uncertainties can combine in the final answer:
  1. Example: if the floor of a room is measured at 14.12 feet long and 9.8 feet wide, our calculators would give us an area of \[\text{Area} = 14.12 \text{ ft} \times 4.8 \text{ ft} = 67.776 \text{ sq. ft}\.  
  2. However, 14.12 ft. \Rightarrow 14.12 \text{ ft.} \pm 0.005 \text{ ft.} \Rightarrow \text{the length could range from 14.115 ft. to 14.125 ft.}\]  
  3. Also, 4.8 ft. \Rightarrow 4.8 \text{ ft.} \pm 0.05 \text{ ft.} \Rightarrow \text{the width could range from 4.75 ft. to 4.85 ft.}\]  
  4. So, the largest possible area under these uncertainties would be 14.125 ft. \times 4.85 ft. = 68.50625 sq. ft.  
  5. The smallest possible area under these uncertainties would be 14.115 ft. \times 4.75 ft. = 67.04625 sq. ft.  
  6. The average of those two areas is 67.77625 sq. ft., which is \pm 0.73 sq. ft. from either extreme.
7. Thinking of uncertainty as ± one-half of a finest increment of measurement, we would round the uncertainty to ±0.5 sq. ft., so we would have to round the answer to 68 sq. ft. to reflect this uncertainty.

8. Or, we could have used the rule; the measurement of 4.8 ft. has the least number of significant digits (two sig. figs.), so the final answer can only have two significant digits.

9. A good way to show the work is: Area = 14.12 ft. × 4.8 ft. = 67.776 sq. ft. ➔ round to 2 sig. figs. ➔ = 68 sq. ft.

- One exception to the rule: numbers without a unit of measurement attached are considered exact numbers that have an infinite number of significant digits. Ignore them when determining the number of sig. figs. in your answer.
  1. Example: 100 × 22.85 cm = 2285 cm

Examples:

1) 2.54 in. × 3 in. = 7.62 in² ➔ round to 1 sig. fig. ➔ = 8 in².

2) 310.2 cm x 51.05 cm x 0.100 cm = 1,583.571 cm³ ➔ round to 3 sig. fig. ➔ = 1,580 cm³.

3) 22.85 ft² / 4 ft. = 5.7125 ft ➔ round to 1 sig. fig. ➔ = 6 ft.

4) 358.9 m³ / 22 m² = 16.313 636 36 m ➔ round to 2 sig. fig. ➔ = 16 m.

5) 75.40 g / 7.1000 mL = 10.619 718 31 g/mL ➔ round to 4 sig. fig. ➔ = 10.62 g/mL.

6) 52.5 mi/hr × 3.8752 hr = 203.448 mi ➔ round to 3 sig. fig. ➔ = 203 mi.

Your Turn!!
Identify the number of significant digits show in each of the following measurements.

1) 400 m

2) 200.0 cm

3) 0.0001 in.

4) 218 lb.

5) 320 kg

6) 635.000 mm

7) 22,000 sec.
8) 5201 sq. ft.

9) 81 sq. m

Perform the following computations using a calculator, write down the calculator answer, then write down the answer rounded to the correct number of significant digits. Your answer must include the unit of measurement to get full credit.

Examples of how to show your work:

7.857 cm + 5.23 cm = 13.087 cm ⇒ 13.09 cm (round to nearest 0.01)
12.14 ft. × 2.2 ft. = 26.708 sq. ft. ⇒ 27 sq. ft. (round to 2 sig figs.)

10) 4.60 cm + 3 cm =

11) 0.008 in. + 0.05 in. =

12) 22.4420 g + 56.981 g =

13) 200 lb. - 87.3 lb. =

14) 67.5 sec - 0.009 sec. =

15) 71.86 hr. - 13.1 hr. =

16) 357.89 in. + 0.002 in. =

17) 17.95 sq. in. + 32.42 sq. in. + 50 sq. in. =
18) \(5.5 \text{ m}^2 + 3.7 \text{ m}^2 + 2.97 \text{ m}^2 =\)

19) \(84.675 \text{ gal} - 3 \text{ gal} =\)

20) \(75 \text{ cu. ft} - 2.55 \text{ cu. ft} =\)

21) \(10 \text{ L} - 9.9 \text{ L} =\)

22) \(13.7 \text{ ft} \times 2.5 \text{ ft} =\)

23) \(13.7 \text{ mL} \times 4 =\)

24) \(200 \times 3.58 \text{ cu. in} =\)

25) \(0.00003 \text{ m} \times 727 \text{ m} =\)

26) \(5003 \text{ ft}^3 / 3.781 \text{ ft} =\)

27) \(89 \text{ ft}^2 / 9.0 \text{ ft} =\)

28) \(5000 \text{ cm} / 55 \text{ cm} =\)

29) \(314 / 100 \times 5.6 \text{ cm} \times 5.6 \text{ cm} =\)
30) 80 mi. / 0.675 hr. =

31) 300 mi/hr. × 10.6 hr. =

32) 0.059 min. × 6.95 mi./min. =

33) 0.003 g / 106 mL =

34) 8.5 g / 0.356 mL =

Section 1.4: Signed Numbers
When we are using numbers to express the change in a quantity, such as the amount of money in a checking account or a running back’s total yards, we soon find that the quantities under study don’t always increase. Bank accounts sometimes decline and running backs can lose yards! To represent a change, which decreases, we use negative numbers, while positive numbers represent an increase. A convenient way to visualize positive and negative numbers is the number line shown below. Here the positive (or “ordinary”) numbers are to the right of zero and the negative numbers are to the left of zero.

The opposite of 5 is –5 since –5 + 5 = 0. For example, if you lose $5 then make $5, you’re back to zero. By the same argument the opposite of –5 is 5. If a running back gains 10 yards, then loses 7, his net yardage is 3. In symbols, 10 + (–7) = 3. So adding a negative 7 is the same as subtracting a positive 7. Also 10 + (–7) = (–7) + 10 = 3. In general, a + (–b) = a – b, i.e., subtracting is the same as adding the opposite and visa versa.

Suppose a running back loses 3 yards every time he carried the ball. If he had four carries, his net yardage is –3 + (–3) + (–3) + (–3) = –12. [You may have noticed that we don’t write + –3, but rather + (–3), this is just to avoid the potential confusion of two adjacent operation symbols.]

However, using the definition of whole number multiplication as repeated addition, we see that 4(–3) = –3 + (–3) + (–3) + (–3) = –12. So a positive number times a negative number should result in a negative number. What about a negative times a negative? One of the fundamental rules of arithmetic is called the distributive property. It says that

\[ a \cdot (b + c) = a \cdot b + a \cdot c \]

For example,
\[ 5 \cdot (4 + 7) = 5 \cdot 4 + 5 \cdot 7 = 20 + 35 \]
\[ 5 \cdot 11 = 55. \]

Now,
\[ (-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1) \]
\[ (-1) \cdot (0) = -1 + (-1) \cdot (-1) \]
\[ 0 = -1 + (-1) \cdot (-1). \]

So \((-1)(-1)\) added to \(-1\) gives zero. But only \(1\) added to \(-1\) makes zero. So we conclude that
\[ (-1)(-1) = 1. \]

In general, a negative number times a negative number gives a positive number. We have analogous statements in English. If I say “I am not dishonest”, the double negative makes the sentence equivalent to saying “I am honest”.

We now have another way of forming the opposite of any number, simply multiply by \(-1\), i.e., \(-b = (-1)b\). The standard order of operations requires that we square before multiplication. This means that \(-5^2 = -1(5^2) = -25\), while \((-5)^2 = (-5)(-5) = 25\). Consider now subtracting a negative number as in \(10 - (-8) = 10 + (-1)(-8) = 10 + 8 = 18\). So subtracting a negative is the same as adding the positive.

Finally, division is the opposite operation to multiplication. Since \((-5)(6) = -30\) and \((-5)(-6) = 30\), then
\[ (-30) \div (-5) = 6 \]
\[ (-30) \div 6 = -5 \]
\[ 30 \div (-5) = -6 \]
\[ 30 \div (-6) = -5. \]

In general, a negative number divided by a negative number is a positive number, while a negative divided by a positive or a positive divided by a negative is negative.

To enter a negative number on some calculators use the minus sign key \([-]\) before the number just as it is written. For example, to evaluate
\[ \frac{6 \times (-10)}{-3 - 2} = \frac{-60}{-5} = 12 \]
enter the following keystrokes : \([6 \times \boxed{-} 10 \div \boxed{-} \boxed{3} \boxed{2} \boxed{=}]\).

The keystrokes for some other calculators are shown below. Note: you enter the negative numbers “backwards”, i.e., first enter the value, then the change sign key.
\([6 \times 10 \boxed{+} \boxed{-} \boxed{3} \boxed{+} \boxed{-} 2 \boxed{=}]\).
Your Turn!!

Perform the indicated operations giving answers to the stated number of decimal places:

7) \( \sqrt{0.144} \) (three places)

8) \( 2.53^2 \times 1.96 - 5.36^2 \div 2.89 \) (two places)

9) \( \sqrt{17^2 - 13^2} \) (one place)

10) \( \frac{3}{7} \times 8 - 1 \) (two places)

11) \( 11.17^2 \times 5.10 - 5.97^3 \div 2.17 \) (one place)

Write the following fractions as decimals:

12) \( 2 \frac{11}{16} \)

13) \( 1 \frac{7}{12} \)

Calculate the following:

16) \( -56 \div 8 \)

17) \( (-8)^2 \div (-16) \)
\[-8^2 \div (-16)\]  
\[
\frac{(-5)(-4)}{-10}
\]

\[
\frac{(-2 \frac{3}{5})(-3)}{-6}
\]

In the first six months of the year, Precision Auto Body had the following profit and loss record:

<table>
<thead>
<tr>
<th>Month</th>
<th>Profit/Loss</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>profit</td>
<td>$8,736.52</td>
</tr>
<tr>
<td>February</td>
<td>profit</td>
<td>$12,567.34</td>
</tr>
<tr>
<td>March</td>
<td>loss</td>
<td>$1,282.72</td>
</tr>
<tr>
<td>April</td>
<td>profit</td>
<td>$478.68</td>
</tr>
<tr>
<td>May</td>
<td>loss</td>
<td>$179.66</td>
</tr>
<tr>
<td>June</td>
<td>profit</td>
<td>$1,257.23</td>
</tr>
</tbody>
</table>

Find the total profit or loss for this six month period. 

24) ________________

Additional Drill:

1) \[-94.82 - (53.96) =\]

2) \[-99.44 - (72.59) =\]

3) \[76.45 + (-73.19) =\]

4) \[62.58 - (-83.61) =\]

5) \[-0.96 + (5.8) =\]

6) \[64.62 - (-14.84) =\]

7) \[33.35 - (-66.95) =\]
8) \(-52.21 + (-55.73) = \)

9) \(15.23 + (-66.09) = \)

10) \(2.16 - (-50.81) = \)

11) \(85.31 - (-30.36) = \)

12) \(-2.31 - (-57.8) = \)

13) \(-80.15 - (-55.23) = \)

14) \(70.57 + (-45.95) = \)

15) \(-77.61 + (7.75) = \)

16) \(-39.67 + (-80.48) = \)

17) \(33.3 - (8.89) = \)

18) \(-50.7 - (-83.65) = \)

19) \(-5.59 - (81.11) = \)

20) \(7.31 + (-90.69) = \)

21) \(3.76 \times (5.97) = \)

22) \(-99.22 \div (-56.23) = \)

23) \(-52.34 \div (83.08) = \)
24) \(-5.78 / (49.01) = \)

25) \(-64.7 / (77.96) = \)

26) \(-6.22 \times (-63.77) = \)

27) \(-51.65 / (22.65) = \)

28) \(-35.88 \times (83.51) = \)

29) \(-1.68 / (-10.43) = \)

30) \(-64.36 / (-13.11) = \)

31) \(-95.62 \times (65.71) = \)

32) \(-52.47 / (-40.78) = \)

33) \(-84.93 / (26.04) = \)

34) \(-4.58 \times (-81) = \)

35) \(12.25 \times (61.62) = \)

36) \(31.28 / (-10.61) = \)

37) \(21.25 / (-49.67) = \)

38) \(18.82 \times (60.86) = \)

39) \(-31.5 \times (10.9) = \)
40) -85.59 x (88.39) =

**Section 1.5: Exponents**

**Section 1.5.1: Definition of Positive Integer Exponents**

Exponents were originally developed as a shortcut notation for repeated multiplication:

\[ 2 \cdot 2 \cdot 2 = 2^3 \]

Exponents are used so frequently in variable expressions that it will be useful to have rules for several common situations. Although every exponent expression can be simplified by multiplying copies of the base, it is easier and quicker to apply the laws of exponents when appropriate. Each exponent law below is a consequence of the definition of powers: \( a^n \) represents the product of \( n \) copies of the base \( a \). If you are ever unsure about which rule to apply, come back to this definition.

**Section 1.5.2 The Meaning of Zero and Negative Exponents**

If we look at the pattern of repeatedly dividing a number that is a perfect power by the base, we can see a meaningful value for a base raised to a power of zero, and for a base raised to a negative power:

\[
\begin{align*}
16 &= 2^4 \\
8 &= 2^3 \\
4 &= 2^2 \\
2 &= 2^1 \\
1 &= 2^0 \\
\frac{1}{2} &= 2^{-1} \\
\frac{1}{4} &= 2^{-2} \\
\frac{1}{8} &= 2^{-3} \\
\frac{1}{16} &= 2^{-4}
\end{align*}
\]

Any base (except a base of zero) raised to a power of zero is equal to one. Here is another explanation:

\[
\begin{align*}
x^3 &= 1xx \quad (1 \text{ times } 3 \text{ factors of } x) \\
x^2 &= 1xx \quad (1 \text{ times } 2 \text{ factors of } x) \\
x^1 &= 1x \quad (1 \text{ times } 1 \text{ factors of } x) \\
x^0 &= 1 \quad (1 \text{ times no factors of } x)
\end{align*}
\]

Any base raised to a negative power is the same as that base raised to the positive of that power, but on the other side of the fraction bar.
Simplify expressions of the form \(a^m \cdot a^n\)
This form applies any time we have a **product of two numbers with the same base**. Since the exponent counts the number of copies of the base, we need to find the total number of copies in the product. Note that \(x^5 \cdot x^2\) means \((xxxxx)(xx) = xxxxxxx = x^7\). We can simplify the product of same bases by adding the exponents together. Normally, we do not write out the long string of variables (it is only done here to illustrate the idea). As a general rule, \(a^m \cdot a^n = a^{m+n}\).

**Examples**

\[ a^3 \cdot a^9 = a^{3+9} = a^{12} \]
\[ y \cdot y^4 = y^{1+4} = y^5 \]
\[ c^3(c^{10})(c^5) = c^{3+10+5} = c^{18} \]
\[ (-5b^3)(7b^8) = -35b^{11} \]

In short, we can multiply the coefficients and add the exponents to multiply on the same base.

Simplify expressions of the form \((ab)^n\)
This rule covers any **power over a product**. Technically, the product inside the parentheses should be completed first, but this may not be possible when variables are involved. However, exponents are compatible with multiplication, so we can expand the power and collect the similar factors together. For example, \((ab)^3\) means \((ab)(ab)(ab)\), which can be regrouped as \(aaabbb\) (the order of multiplication does not affect the product). In short, there are 3 \(a\)'s and 3 \(b\)'s in the product, so \((ab)^3 = a^3b^3\).

In general, the rule for a power over a product is \((ab)^n = a^nb^n\).

**Examples**

\[(xy)^5 = x^5y^5 \]
\[(2pq)^4 = 2^4p^4q^4 = 16p^4q^4 \]
\[ (-3t)^2 \text{ means } (-3t)(-3t) = +9t^2 \]

**Note:** The parentheses make a big difference in the calculations because they change the normal priorities for the order of operations. If the last example above had no parentheses, we get a different result: \(-3t^2\) does not simplify because the exponent applies to the variable only (do exponents before multiplications).
Simplify expressions of the form \((a^m)^n\)

This form shows a **single base with successive powers**. \((a^m)^n\) represents the product of \(n\) copies of \((a^m)\). That is, we need \(m\) copies of the base \(n\) times over. The result is \((a^m)^n = a^{m\cdot n}\). For an example, consider \((x^5)^2\). This represents the product of 2 copies of \((x^5)\). Thus \((x^5)^2 = (x^5)(x^5) = x^{10}\) (note how the first law of exponents applies, because now we are multiplying two expressions with the same base \(x\)). The shortcut is to simply multiply the powers together to get the simplified form of the answer, \((x^5)^2 = x^{5\cdot 2} = x^{10}\).

**Examples**

\[
(y^7)^3 = y^7 \cdot 3 = y^{21}
\]

\[
(5^3)^6 = 5^{3\cdot 6} = 5^{12}
\]

\[
(z^9)^5 = z^{45}
\]

Simplify expressions involving two or more of these forms

Some expressions will involve more than one of the exponent laws. You will need to carefully determine which laws to apply. In general, start by simplifying within any parentheses groups. Work your way outward, applying exponents to the factors of each group, and then finish by multiplying the simplified groups together.

**Example**

Simplify \((3x^3y^4)^2\).

**Solution**

This is a combination of the second and third forms: \((ab)^n = a^n b^n\) and \((a^m)^n = a^{m\cdot n}\). The power 2 applies to each factor inside. Since some of these factors already have powers, the exponents are multiplied. Thus \((3x^3y^4)^2 = 3^2(x^3)^2(y^4)^2 = 9x^6y^8\).

**Example**

Simplify \((a^2b)^3(2a^4)\).

**Solution**

Simplify the first group by applying the exponent 3 to each factor.

\[
(a^2b)^3(2a^4) = (a^6b^3)(2a^4).
\]

Then multiply the result by \((2a^4)\) to get the final answer, \(2a^{10}b^3\).

**Example**

Simplify \((5p^4q^3)^2(3pq)^2\).

**Solution**

Each group must be simplified first, using the exponent on the right parenthesis.

\[
(5p^4q^3)^2(3pq)^2 = (25p^8q^6)(9p^2q^2) = 225p^{10}q^8.
\]

The following table summarizes the exponent rules introduced in this section. Use each rule only if the form of the expression exactly matches. The exponents \(n\) and \(m\) may be any numbers. The variables \(x\) and \(y\) are used generically and could be any letter or number in the base.
Rules of Exponents

\[
x^n \cdot x^m = x^{n+m} \quad (x \cdot y)^n = x^n y^n \quad (x^m)^n = x^{m \cdot n}
\]

Your Turn!!

Simplify the following exponent expressions.

1. \(2^2 \cdot 2^5\)
2. \((3^3)^2\)
3. \(10^7 \cdot 10^{11}\)
4. \(x^3(x^7)\)
5. \((x^3)^7\)
6. \(x^2 \cdot x^6\)
7. \(y \cdot y^5 \cdot y^3\)
8. \((y^3)^3\)
9. \((b^9)^2\)
10. \((2x^3)^4\)
11. \((3c^4)^2\)
12. \((ab)^3\)
13. \((10^4 \cdot 10^2)^2\)
14. \((x^3)^4\)
15. \((-5x^2y)^2\)
16. \((10^4)^3 \cdot 10^5\)
17. \((-3x^2y)^3\)
18. \((p^5)(p^8)\)
19. \(x^5(-x^3)^2\)
20. \((4x^3y^5)^2 \cdot (3xy^2z)\)
21. \((8h^2k^5)(-5hk)\)
22. \((-3a^2b)(10a^4b^3)\)
23. \((3ab^2)(3a^2b)(3ab)^2\)
24. \(-(-5a^3b^4)^3\)
Section 1.5.3 Quotient Rule and Negative Exponents

The exponent rules in the previous section cover simplification of variable expressions involving multiplication only. The division operation is closely related to multiplication, and we can also simplify quotients by applying and expanding on the exponent rules.

Simplify exponential expressions using the property \(\frac{a^m}{a^n} = a^{m-n}\)

According to the multiplicative property for exponents, \(a^m \cdot a^n = a^{m+n}\). To multiply numbers with the same base, we add the exponents. Since division is the opposite of multiplication, it makes sense to guess that the rule uses subtraction of the exponents instead. In reality, we are reducing out common factors in the division problem.

Example Simplify \(\frac{x^5}{x^3}\).

Solution Write each exponent form out the long way as a repeated multiplication.

\[
\frac{x^5}{x^3} = \frac{xxxxx}{xxx} = \frac{xx}{1} = x^2.
\]

The common factors of \(x\) reduce out since each \(x/x\) is 1. Using the rule, we get the same answer, but more efficiently: \(\frac{x^5}{x^3} = x^{5-3} = x^2\).

So to divide expressions with the same variable base, we subtract the exponents. The quotient property can be applied to both numeric and variable expressions, as long as the numerator and denominator have powers of the same base.
Examples

\[
\frac{2^7}{2^4} = 2^{7-4} = 2^3 = 8
\]

\[
\frac{w^{10}}{w} = w^{10-1} = w^9
\]

Simplify expressions with exponent zero

The quotient rule for exponents introduces a couple special cases that we will need to understand in order to fully utilize the properties. The first comes up when the numerator and denominator have the same exponent. After subtracting, we will have a zero exponent.

Example  Simplify \( \frac{x^3}{x^7} \).

Solution  Apply the quotient rule and subtract the exponents:

\[
\frac{x^3}{x^7} = x^{3-7} = x^{-4}
\]

But what does \( x^0 \) really mean? The answer lies in the reduced form of the fraction.

Alternate Solution  Reduce the common factors:

\[
\frac{x^3}{x^3} = \frac{xxx}{xxx} = \frac{\cancel{x} \cancel{x} \cancel{x}}{\cancel{x} \cancel{x} \cancel{x}} = \frac{1}{1} = 1.
\]

Comparing the two solutions, we must have \( x^0 = 1 \) for any value of \( x \) except 0 (we will not worry about that case until a Calculus class). In general, a 0 exponent means we multiply no copies of the base. Since all multiplication starts with 1, and we multiply nothing else, the answer must be 1. If you are still not convinced, look at the pattern for decreasing exponents:

\[
\begin{align*}
x^3 &= 1xxx \quad (1 \text{ times 3 copies of } x) \\
x^2 &= 1xx \quad (1 \text{ times 2 copies of } x) \\
x^1 &= 1x \quad (1 \text{ times 1 copy of } x) \\
x^0 &= 1 \quad (1 \text{ times no copies of } x)
\end{align*}
\]

Although we usually do not write the coefficient 1, it is still there and must be written when there is nothing else.

Examples  \( w^0 = 1 \)

\( 26^0 = 1 \)

\( (5b)^0 = 1 \)

Note that the 0 power applies to the whole group.
$5b^0 = 5 \cdot 1 = 5$
Here we do have a coefficient (namely 5), but no copies of the variable.

$(3x)^0 - 10x^0 = 1 - 10(1) = 1 - 10 = -9$

**Simplify expressions with negative exponent**
The second case involves **negative exponents**. If the denominator has a higher power than the numerator, we get a negative exponent after subtracting.

**Example**
Simplify $\frac{x^3}{x^8}$.

**Solution**
Apply the quotient rule and subtract the exponents:

$$\frac{x^3}{x^8} = x^{3-8} = x^{-5}.$$  

**Alternate Solution**
Reduce the common factors:

$$\frac{x^3}{x^8} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^5}.$$  

The negative exponent really indicates that we have a fraction with more division than multiplication.

Comparing the two solutions, we see that $x^{-5} = \frac{1}{x^5}$. The negative power refers to a **division** by repeated copies of the variable. In general, $x^{-n} = \frac{1}{x^n}$; replace the negative power by using 1 over the variable with a positive power.

**Example**
Simplify $\frac{c^3}{c^5}$.

**Solution**
Apply the quotient rule and subtract the exponents:

$$\frac{c^3}{c^5} = c^{3-5} = c^{-2}.$$  

The negative exponent can also be replaced by making a fraction expression:

$$\frac{c^3}{c^5} = c^{-2} = \frac{1}{c^2}.$$
Generally, the fraction form with positive exponent is preferred, but there are times that the negative exponent is more convenient.

Examples

\[
3^{-2} = \frac{1}{3^2} = \frac{1}{9} \quad \quad \quad 8^{-1} = \frac{1}{8^1} = \frac{1}{8}
\]

\[
w^{-4} = \frac{1}{w^4}
\]

\[
(4y)^{-2} = 4^{-2}y^{-2} = \frac{1}{4^2} \cdot \frac{1}{y^2} = \frac{1}{16y^2}
\]

\[
x^3 \cdot x^{-7} = x^{3-7} = x^{-4} = \frac{1}{x^4}
\]

\[
\frac{p^{-5}}{p^{-2}} = p^{-5-(-2)} = p^{-3} = \frac{1}{p^3}
\]

In summary, positive exponents refer to a multiplication by copies of the base. The number of copies is given by the exponent.

With a zero exponent, there is nothing to multiply except the coefficient (which may be 1). In fact, any base with a zero power will be equal to 1. Be careful if there are other coefficients. The power 0 will wipe out any part of the base inside parentheses, but coefficients outside the parentheses will remain.

Negative exponents refer to a division by copies of the base. You may think of the negative sign in the exponent becoming the fraction bar if you like. The negative does not really disappear; it only has a different role to serve.

\[
\begin{array}{cccc}
\text{Rules of Exponents} \\
\hline
x^n \cdot x^m = x^{n+m} & \frac{x^n}{x^m} = x^{n-m} & (x \cdot y)^n = x^n y^n \\
\hline
\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} & \left(x^m\right)^n = x^{mn} & x^{-n} = \frac{1}{x^n}
\end{array}
\]

Section 1.5.4 Scientific notation
Exponents can be used to write very large and very small numbers in a more concise way. Scientific notation is commonly used in many science applications. Astronomy, for example, deals with very large scales, while biology and chemistry often study objects of very small sizes. This section introduces the form of the scientific notation and provides some practice in converting numbers between standard forms and scientific notation. Scientific notation is considered simpler because there is no need to write a long string of 0 digits at the end of a number.

Change numbers written in scientific notation to standard (decimal) form
A typical number written in scientific notation looks like $3.4 \times 10^5$. The number has a decimal part (3.4, in this case) that is multiplied by a power of 10. We can convert back to standard form simply by multiplying out the product. The exponent is done first, of course:

$$10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000.$$ 

A positive power of 10 always gives a 1 followed by a number of zeroes equal to the exponent. Now multiply this by the decimal part $3.4 \times 100,000$ is equal to 340,000. Note that it is standard to use “$\times$” for “times” in scientific notation. This is perhaps the only place we will routinely use the “$\times$” symbol in algebra.

Multiplying any number by 10 simply shifts the decimal point one place to the right. For powers of 10, the decimal point is shifted to the right as many places as the value of the exponent.

**Example**  
$1.52 \times 10^3$ is equivalent to 1,520.  
The decimal point was shifted exactly 3 places to the right. Insert zeros to fill all empty places.

Small numbers (close to zero) are written with negative exponents on the base 10. Since a negative power means we need to divide by the factors of 10, the decimal place is shifted to the left instead.

$$9 \times 10^{-4} = \frac{9}{10^4} = \frac{9}{10,000} = 0.0009.$$ The decimal place was shifted 4 places to the left.

**Example**  
$2.3 \times 10^{-6}$ becomes 0.0000023.  
Note that the decimal point was shifted exactly 6 places to the left.

**Write number in scientific notation**

To write a number in scientific notation, move the decimal place far enough that there is exactly one (nonzero) digit left of the decimal point. Then multiply by a power of 10, where the exponent is the number of places you moved the decimal point (use a positive exponent if the decimal moved left, and a negative exponent if the decimal moved right). This works to represent the same number, because multiplying by a power of 10 simply moves the decimal point back to where it belongs.

**Examples**  
45,000,000 is the same as $4.5 \times 10^7$  
$7,810,000,000 = 7.81 \times 10^9$  
$540 = 5.4 \times 10^2$

**Examples**  
$0.0000003 = 3 \times 10^{-7}$  
(Negative exponents are necessary to write numbers less than 1)  
$0.000764 = 7.64 \times 10^{-4}$

**Multiply and divide numbers using scientific notation**

Using scientific notation can simplify calculations involving large numbers. First, convert numbers to scientific notation, and then use exponent rules to reduce the powers of 10.

**Example**  
Simplify $(320,000)(50,000,000)$. 

Madison College’s College Mathematics Textbook
Solution \( (320,000)(50,000,000) \)
\[ = (3.2 \times 10^5)(5 \times 10^7) \quad \text{Convert to scientific notation} \]
\[ = (3.2)(5) \times 10^5 \times 10^7 \quad \text{Multiply coefficients} \]
\[ = 16 \times 10^{12} \quad \text{Add exponents on the base 10} \]
\[ = 16,000,000,000,000 \quad \text{Move the decimal point back 12 places} \]

Your Turn!!

Simplify the following exponent expressions.

1. \( w^0x^2 \)
2. \( 8w^0x^2 \)
3. \( -9x^0 \)
4. \( 10^{-5} \cdot 10^9 \)
5. \( (15d)^0 \)
6. \( 5x^0 + (9x)^0 \)
7. \( x^{-5} \)
8. \( 7t^{-4} \)
9. \( (4b)^{-3} \)
10. \( 10^4 \cdot (10^{-3})^3 \)
11. \( \left( \frac{2 \times 10^{-4}}{10^{-6}} \right)^2 \)
12. \( \frac{12xy^2}{4xy^{-1}} \)
13. \( (2a^{-5})^{-2} \)
14. \( (5x)^{-4} \)
15. \( \frac{12xy^2}{4xy^{-3}} \)
16. \( (3x^2y)^{-3} \)
17. \( \left( \frac{xy^2}{z} \right)^{-1} \)
18. \( \frac{24x^4y^{-3}}{4x^{-4}y^3} \)

Write the following numbers in standard (decimal) form.

19. \( 1.7 \times 10^8 \)
20. \( 9.17 \times 10^{-5} \)
21. \( 4.23 \times 10^6 \)
22. \( 2.05 \times 10^{-7} \)

Write the following numbers in scientific notation.
Section 1.6: Order of Operations

Mathematical expressions which involve more than one operation appear ambiguous. For example, is

\[ 5 + 6 \times 2 = 11 \times 2 = 22 \quad \text{or} \quad 5 + 6 \times 2 = 5 + 12 = 17 \ ? \]

To clarify this question, mathematics has developed the following hierarchy of computations called order of operations.

1. Perform all operations that appear in grouping symbols first. If grouping symbols are nested, do the innermost first.
2. Raise all bases to powers in the order encountered moving from left to right.
3. Perform all multiplications/divisions in the order encountered moving from left to right.
4. Perform all additions/subtractions in the order encountered moving from left to right.

Here grouping symbols means parentheses ( ), brackets [ ], braces { }, etc. An example of a nested expression is

\[ (6 + 2 \times (4 + 1)) + (6 \times 5) + (6 + 10) \div 8 = 16 + 8 = 2. \]

Raising a base to a power (also known as an exponent) means repeated multiplication as in \(3^6 = 6 \times 6 \times 6 = 216\). To perform this calculation on some calculators enter the following keystrokes: \(6 \uparrow 3 \uparrow \). The \(\uparrow\) key is called a carrot key. If your calculator does not have the \(\uparrow\) carrot key, it probably has the \(y^x\) exponent key. To perform this same calculation on this type of calculator enter the following keystrokes: \(6 \uparrow^3 \times 3 \uparrow\). We will cover exponents in greater detail in Section 1.4 - Decimal Fractions.

In the original problem posed above the multiplication of 6 with 2 is performed before the addition of 5. The proper answer is therefore 17. The other interpretation could be achieved by using parentheses \((5 + 6) \times 2 = 11 \times 2 = 22\). Order of operations is built into all scientific calculators. That is, if you enter the keystrokes in the correct order, the calculator will automatically perform the correct calculation. Therefore \(5 \div 6 \times 2 \div\) yields the correct answer of 17. Try your own calculator to see if you get the correct answer. Then you will know if your calculator performs the order of operations correctly or not.

In many formulas \(x\) occurs as a variable, but then confusion with the times sign can result. To avoid this, alternative symbols for multiplication are used. They are the dot notation and adjacent parentheses as in \(7 \times 3 = 7 \cdot 3 = (7)(3) = 21\). Some calculators recognize the adjacent parentheses as multiplication, but some do not. On these calculators the times operation \(\times\) must be inserted between the parentheses.
Division is also indicated by a variety of notations. For example, the following all mean 34 divided by 17,
\[ 34 \div 17 = 34/17 = \frac{34}{17} = \left[\frac{34}{17}\right] = 2 \]

In addition to parentheses, brackets and braces, certain symbols act as **implied grouping symbols**. The most important of these are the fraction bar and the square root symbol.
The fraction bar acts to separate the numerator from the denominator. If either or both of the numerator or denominator consist of an expression with operations, these must be performed first before the division indicated by the fraction bar. For example,
\[ \frac{7+3}{2+3} = \frac{10}{5} = 2 \]
To perform this computation on the calculator, parentheses need to be inserted around both numerator and denominator as shown below.

\[
\left( \frac{7}{2} + \frac{3}{3} \right) = \left( \frac{2}{2} + \frac{3}{3} \right)
\]

Parentheses are the only grouping symbol the calculator recognizes or uses.
The square root symbol also acts as a grouping symbol. Any calculation inside the square root needs to be completed before the root is taken. For example, \( \sqrt{25 + 144} = \sqrt{169} = 13 \).

To perform this computation on the calculator parentheses need to be inserted around the expression inside the square root symbol.

On some calculators enter the following keystrokes : \( \sqrt{\left( 25 + 144 \right)} \)

On other calculators enter the keystrokes: \( \left( 25 + 144 \right) \sqrt{} \)

Note: On “newer” calculators like the Casio fx-300MS one enters expressions the way “they look”, i.e., the square root symbol comes first. On older models like the TI-30Xa some functions like \( \sqrt{} \) come after the expressions they are to evaluate. Give the above exercise a try to find out how your calculator works. In any case, using parentheses keys when necessary is a good habit to acquire. Failure to do so usually results in wrong answers!

**Your Turn!!**

Perform the following arithmetical operations. Remember to follow the correct order of operations.

1. \( 12 \times 9 - 8 \times 7 \div 4 \times 2 = \) \[ \text{________} \] 2. \( 12 \times 9 - 8 \times 7 \div (4 \times 2) = \) \[ \text{________} \]

3. \( 4(19-12) \div 2-11 = \) \[ \text{________} \] 4. \( 32 - 3 \times 9 + 16 \div 4 \times 2 = \) \[ \text{________} \]
5. \[ \frac{561}{51} \] = ________ 6. \[ 12 + \frac{18}{3} - 5 \] = ________ 

7. \[ \frac{5 + 13 + 2}{1 + 3} \] = ________ 8. \[ \frac{12 \times 5 \times 4}{6 \times 2 + 3} \] = ________ 

9. \[ \sqrt{3^2 + 4^2} \] = ________ 10. \[ 4[3 + 2(9-2)] \] = ________ 

11. \[ 3^3 [ (6 + 5) - 2^2 ] \] = ________ 12. \[ \frac{4 \cdot 10 + 9 \cdot 2}{2(4 - 2)} \] = ________ 

From a board 10 feet long a piece 29 inches was cut off. How long is the piece remaining? Ignore the width of the cut.

13) ________________

In 1993 General Motors had total sales of $133,621,900,000, while Ford Motor Company had sales of $108,521,000,000. How much more were GM’s sales than Ford’s?

14) ________________

If 128 copies of a software package cost $5760, what is the cost per copy?

15) ________________

A carpenter earns $14 per hour plus time and a half for overtime (more than 40 hours in one week). If she works 58 hours one week, what is her gross pay for that week?

16) ________________

You order 15 CD’s at $8 a CD and 24 cassette tapes at $3 a tape. What is the cost of the order?

17) ________________

A truck averages 16 miles per gallon and has a 25 gallon gas tank. What is the furthest distance the truck can travel without stopping for gas?

18) ________________
A stairway consists of rises of 6 in and must reach a height of 10 feet. How many rises are needed?

19) ________________

A stairway consists of 4 in rises and treads of 18 in. If the height of the stairs is 4 feet, what is the distance taken up by the stairway on the lower floor?

20) ________________

Section 1.7: Evaluating and Simplifying Expressions

Algebra, despite its fearsome reputation, is really just "generalized" arithmetic. The name itself comes from the title of a ninth century Arabic mathematics text *Ilm al-jabr wa'l mugabalah* written by Al-Khowarozmi in the court of Al-Maman, the Caliph of Bagdad. Loosely translated, this is the art of working backwards to reach a solution. The modern word “algebra” has come to mean any of the mathematics involving the use of letters to symbolize an unknown number.

Basic vocabulary

Any letter used to represent a number value is called a **variable**. The letter $x$ is the most commonly used variable, but a variable can be any letter of the alphabet, capital or lower case, or even other symbols. Variables are used in place of actual numbers when the true value is unknown or when the number may change. Algebra is very useful for analyzing situations where two or more quantities depend on one another.

An **operation** is any mathematical instruction between two numbers or variables that calls for a specific calculation. The basic mathematical operations are addition, subtraction, multiplication, and division.

An **expression** is a meaningful collection of numbers, variables, and operations. “$27 - 10$” is a numeric expression because it has no variables, just a couple numbers and the subtraction operation. “$9a - 5b$” is a variable expression, because it has variables ($a$ and $b$) as well as numbers (9 and 5) with operations (multiplication and subtraction).
Three algebraic concepts: Evaluate, Simplify, and Solve
All of the work in algebra can be classified as one of three main tasks to perform.

**Evaluate** means to use a given or known value for the variable to find a single numerical value for the answer. We can replace variables with actual values when they are known. A common use of evaluation is to replace letters in a formula with specific numbers to determine an answer. For example, the formula $C = \frac{5}{9}(F - 32)$ can be used to convert a Fahrenheit temperature $F$ to an equivalent Celsius temperature $C$.

**Simplify** means to rewrite an expression with variables in a simpler, but equivalent form. It is preferable to work with fewer symbols and smaller numbers whenever it is possible to change an expression into a nicer form. For example, $4x + 3x$ can be simplified to $7x$, which is simpler because it does not have the addition symbol, and it has only one use of the variable $x$.

**Solve** means to use information (such as an equation) about a variable to determine its precise value (or all possible values, if more than one answer works). For example, the equation $2x + 1 = 7$ is only true when $x = 3$. To solve the equation means to systematically find this value. The methods needed to reduce the original equation to the solution value make up the Algebra that Al-Khowarozmi was beginning to develop in his historic text.

The first and most important step in evaluation is to follow the universal **order of operations**. This is a set of rules for making any series of mathematical calculations. When we have more than one operation in an expression, we must agree to follow the same order of operations, or we will never agree on the answers. All mathematics, worldwide, follows these same rules of order. Scientific and graphing calculators are also programmed to follow the correct order.

The order of operations can be summarized with the acronym “PEMDAS:”

P) Work inside the Parentheses first. Parentheses are used to give priority to some part of the whole expression. Parentheses include grouping symbols of all shapes: ( ), [ ], or { }.

E) Simplify all Exponent calculations next.

MD) Multiply and Divide, working from left to right if there are more than one of these.

AS) Add and Subtract any remaining terms, working left to right.

**Evaluate a numeric expression**
Follow the order of operations to simplify any expression involving numbers and operations to a single value.

**Example** Evaluate the expression $17 - 5 \cdot 2$.

**Solution** Multiplication is higher than subtraction in the order priority, so we must multiply $5 \cdot 2$ first and replace it with the result 10. The subtraction is done last.

$$17 - 5 \cdot 2 = 17 - 10 = 7$$
Note: The middle dot between two numbers indicates a multiplication, not a decimal point. The times symbol × is avoided in algebra because it is too easy to confuse with the variable x. It is also customary to drop the symbol entirely, so whenever there is no symbol between two numbers, the operation is multiplication.

The expression $17 - 5 \cdot 2$ can also be written as $17 - 5(2)$. Here, there is no symbol between the number 5 and the number (2), so multiplication is assumed.

**Example** Evaluate the expression $3(5 + 4)$.

**Solution** In this case, the group in parentheses must be calculated first. Replace any expression inside parentheses with its answer before continuing with any operations outside the parentheses.

$$3(5 + 4) = 3(9) = 27$$

**Example** Evaluate the expression $5 + 11(8 - 5) - 16$.

**Solution** Again, the group in parentheses must be calculated first. Then do the multiplication, and finish with the addition and subtraction.

$$5 + 11(8 - 5) - 16$$
$$= 5 + 11(3) - 16$$
$$= 5 + 33 - 16$$
$$= 38 - 16$$
$$= 22$$

**Note:** Addition and subtraction will always be the last operations performed, unless they are grouped inside parentheses.

**Example** Evaluate the expression $8 + (5 - 2^2)$.

**Solution** Again, the group in parentheses must be calculated first.

$$8 + (5 - 2^2) = 8 + (3)^2$$

The raised “2” here is an **exponent**. Exponents are a short-hand for repeated multiplications. $(3)^2$ means to multiply two copies of the base number 3: $(3)(3) = 9$. Replace the exponent calculation $(3)^2$ with the result 9, and then continue following PEMDAS:

$$8 + (5 - 2^2)$$
$$= 8 + (3)^2$$
$$= 8 + 9$$
$$= 17$$
Evaluate a variable expression

With a variable expression, individual variables may be assigned specific values. In this case, replace each letter with the known value, and then follow the order of operations to simplify the remaining numeric expression to a single value.

Example  Evaluate the expression \( a^2 - 2b \) when \( a = 5 \) and \( b = 3 \).

Solution  Think of the variables as place holders. The best way to do this is to replace each letter with empty parentheses. This makes room for the actual value, which then fills in the parentheses. After filling in the given values for \( a \) and \( b \), the exponent calculation has the highest priority, followed by the multiplication, and then the subtraction last.

\[
\begin{align*}
  a^2 - 2b & \quad = \quad (5)^2 - 2(3) \\
          & \quad = \quad 25 - 6 \\
          & \quad = \quad 19 \\
\end{align*}
\]

Example  Evaluate the expression \( y^2 - 2(x + 7) \) when \( x = 4 \) and \( y = 9 \).

Solution  Replace each variable with the known value and then follow PEMDAS.

\[
\begin{align*}
  y^2 - 2(x + 7) & \quad = \quad (9)^2 - 2(4 + 7) \\
                  & \quad = \quad 81 - 22 \\
                  & \quad = \quad 59 \\
\end{align*}
\]

Your Turn!!

Evaluate the following numeric expressions:

1. \( 33 - 17 \)  
2. \( -(33 - 17) \)
3. \( 7 + 3 - 6 \)  
4. \( 4 \cdot 3 - 8 \)
5. \( 7 - 2(3) \)  
6. \( 5 + 3^2 \)
7. \( 8(5 - 3) - 3^2 \)  
8. \( 4(9 - 6) \)
9. \( 2 + 3(7 - 3) \)  
10. \( 5(2 + 3) - 3 \cdot 4^2 \)
11. \[ 5^2 - 3(5) + 1 \]
12. \[ (-4)^2 - 3(-4) + 1 \]
13. \[ 1.56 - 7.12 + 2.15^2 \]
14. \[ -7 - \frac{12}{3}(-4)^3 \]
15. \[ \frac{235.1 - 142.6(1.73)^2}{38.6 + 1.8^3(2.13)^2} \] (Round to two decimal places)
16. \[ -6^2 - (-2)^3 \]

Evaluate the following variable expressions with the given values:

19. \( a^2 - b \) when \( a = 2 \) and \( b = 3 \)
20. \( 4x^2 \) when \( x = 5 \)
21. \( 3a^2 + 7a - 12 \) when \( a = 2 \)
22. \( 2 + 3(w - x)^2 \) when \( w = 4 \) and \( x = 1 \)
23. \( k^2 + 4k - 1 \) when \( k = -3 \)
24. \( s^2t + 3r \) when \( s = 5 \), \( t = 2 \), and \( r = 4 \)
25. \( (x + 8y - xz)^2 \) when \( x = 7 \), \( y = 1 \), and \( z = 2 \)
26. \( \frac{1}{2}bh \) when \( b = 14 \) and \( h = 11 \).
27. \( \frac{5}{9}(F - 32) \) when \( F = 77 \)
28. \( \frac{h(a + b)}{2} \) when \( a = 5.5 \), \( b = 2.5 \), and \( h = 6.5 \).

To simplify variable expressions, we need to learn what happens with different operations. Naturally, addition has a different effect than multiplication. In this section, we will look at some basic simplification techniques involving addition, subtraction, and multiplication in variable expressions.

**Addition and subtraction of variable terms**

Even if you have never worked with variables, you are probably already familiar with combining like terms. For example, the sum of 4 boxes and 5 boxes is 9 boxes. You simply add \( (4 + 5 = 9) \) and keep the “boxes” as the like units. Now let \( x \) be a variable that represents one box. The calculation becomes \( 4x + 5x = 9x \). In words, you could say “4 \( x \)’s and 5 \( x \)’s make 9 \( x \)’s. This works for any variable: \( 7a - 20a = -13a \), and \( -2b - 3b = -5b \).

The number in front of any variable is called the **coefficient** of the variable. The coefficient is the number attached a variable term by multiplication. To add or subtract variable terms, we combine the coefficients and keep the same variable (these are called **like terms**).
Note that we usually do not write a coefficient of 1, so \( x \) alone really means \( 1x \). Thus \( 9x + x = 10x \). This is consistent with our English language, where we might refer to “a box” without emphasizing that we specifically mean “one box.”

If two terms have different variable parts, then we cannot simplify the expression. For example, the expression \( 12b + 4w \) has no simpler form. How else could you write “12 bananas and 4 watermelons” without losing information? Also note that a constant term (one with no variable) is different from any variable term. If terms have different variables, leave them alone. If they have the same variable, you may combine the coefficients.

**Examples**

6\( y \) + 10\( y \) simplifies to 16\( y \)

3\( t \) – 21\( t \) simplifies to –18\( t \)

62\( w \) – 70\( w \) simplifies to –8\( w \)

–20 + 4\( x \) – 3 simplifies to 4\( x \) – 23

8\( x \) – 3 + 7\( x \) + 9 simplifies to \((8x + 7x) + (-3 + 9) = 15x + 6\)

9\( m \) + 3\( p \) – 4\( m \) + 12 simplifies to 5\( m \) + 3\( p \) + 12 (combine only the two \( m \) terms)

Variable terms must also have the same exponent to qualify as like terms. \( 3x^2 \) and \( 5x^2 \) are like terms and may be added: \( 3x^2 + 5x^2 = 8x^2 \).

**Note:** The exponent is never changed by an addition or subtraction operation.

**Example** Simplify \( 30t^2 + 7t – 12t^2 + 3t + 5 \).

**Solution** Group the like variable terms.

\[
30t^2 + 7t – 12t^2 + 3t + 5 = (30t^2 – 12t^2) + (7t + 3t) + (-8 + 5) = 18t^2 + 10t – 3
\]

**Multiplication using the distributive property**
The distributive property of real numbers shows how multiplication and addition/subtraction interact. For example, we can see that \( 4(2 + 3) = 4(5) = 20 \) if we simply follow the order of operations. On the other hand, we can multiply first provided we multiply through to each term inside the parentheses: \( 4(2 + 3) = 4(2) + 4(3) = 8 + 12 \), which is 20 again. This property always works, no matter what the numbers are.

In general, we can write the distributive property as \( a(b + c) = ab + bc \) for all real numbers \( a \), \( b \), and \( c \). In other words, when we have a multiplier in front of a parentheses group, we can choose to simplify and remove the parentheses provided that we multiply through to each term inside the parentheses. This is especially useful when variables are involved, because we may have no other way to simplify an expression with parentheses.

**Example** \( 2(x + 3) = 2x + 2(3) = 2x + 6 \)

**Examples** \( 5(p – 9) = 5(p) – 5(9) = 5p – 45 \)

\( 3(4c – 2) = 3(4c) – 3(2) = 12c – 6 \)
$7(2x + 1) = 7(2x) + 7(1) = 14x + 7$

With a negative coefficient in front of the parentheses group, all of the signs inside get changed.

**Example**  Simplify $-2(f + 6)$.

**Solution**  Multiply both terms inside the parentheses by $-2$.

\[
-2(f + 6) \\
= -2(f) - 2(6) \\
= -2f - 12
\]

**Note:** Multiplying two numbers with the same sign (both positive or both negative) always has a positive result. Multiplying two numbers with different signs (one positive and one negative) always has a negative result.

**Example**  Simplify $-4(9v - 2)$.

**Solution**  Multiply both terms inside the parentheses by $-4$.

\[
-4(9v - 2) \\
= -4(9v) - 4(-2) \\
= -36v + 8
\]

With more terms inside the parentheses, multiply the coefficient by every term:

**Example**  Simplify $-6(2x - 5y - 10)$.

**Solution**  Multiply all three terms inside the parentheses by $-6$.

\[
-6(2x - 5y - 10) \\
= -6(2x) - 6(-5y) - 6(-10) \\
= -12x + 30y + 60
\]

For expressions with two or more parentheses groups, simplify first by multiplying each group using the Distributive Property, and then combine any like terms from the two groups.

**Example**  Simplify $2(4x + 9) + 3(2x + 1)$.

**Solution**  Distribute the coefficient through each group first.

\[
2(4x + 9) + 3(2x + 1) \\
= 2(4x) + 2(9) + 3(2x) + 3(1) \\
= 8x + 18 + 6x + 3 \\
= 14x + 21
\]
Example  Simplify $2(m - 7) - 5(2m - 3)$.

Solution  Note that the negative coefficient starting the second group changes signs on both of the following terms.

$$2(m - 7) - 5(2m - 3)$$
$$= 2m - 14 - 10m + 15$$
$$= -8m + 1$$

Your Turn!!

Simplify the following by combining like terms:

1. $11b - 2b$
2. $6h + 7h$
3. $4x + 9 - x$
4. $6x + 5 - 2x - 4$
5. $3y - 9x + 10 - y + 7x + 2$
6. $5w + 8t + 18 - 6t - 7 + 2w$
7. $x^2 + 6x + 15 + 3x^2 - 4x - 7$
8. $14 - 2y + 5y^2 - 10y - 2y^2$

Use the Distributive Property to simplify the following expressions:

9. $2(7x + 5)$
10. $6(x - 3)$
11. $4(3g - 2)$
12. $-2(4x + 3)$
13. $10(4a - 3b + c)$
14. $5(1 - x - 3x^2)$
15. $-(2x - 21)$
16. $-8(1 - 2b)$
17. $(9p - 8) + (3p + 2)$
18. $(6y + 5) - (4y - 11)$
19. $3(r + 8) + 5(2r - 3)$
20. $5(2x - 7) + 9(x - 3)$
21. $3(4f + 1) - 2(3f - 5)$
22. $2(x - 1) + 3(4x + 5) - 4(2x - 3)$
23. $(7y^2 - 2y + 1) - (3y^2 + 6y - 8)$
24. $4(3x^2 - 5x + 2) - 7(2x^2 - 6x - 3)$
Chapter 1 Sample Test

No calculators allowed on this test!

1. (__ / 6 points) \[ 10 \frac{3}{16} - 5 \frac{5}{6} = \]

2. (__ / 6 points) \[ 10 \frac{3}{16} \div 5 \frac{5}{6} = \]

3. (__ / 6 points) Convert \( \frac{2}{11} \) to a decimal.
4. (___ / 6 points) Convert 0.168 to a fraction in lowest terms.

5. (___ / 6 points) \(0.47 \times 1.38 = \)

6. (___ / 6 points) \(2.108 \div 0.17 = \)

7. (___ / 4 points) State your answer using the correct number of significant digits: \(130.00 \text{ miles} \div 2.00 \text{ hours} = \)
8. ( ___ / 4 points ) State your answer using the correct number of significant digits:

\[ 0.111 \text{ cm} + 1.11 \text{ cm} + 11.1 \text{ cm} + 111 \text{ cm} = \]

9. ( ___ / 4 points ) State your answer using the correct number of significant digits:

\[ -12.2 \text{ miles} - 13.5 \text{ miles} = \]

10. ( ___ / 4 points ) State your answer using the correct number of significant digits:

\[ 37.65 \text{ cm} - (-82.95 \text{ cm}) = \]

11. ( ___ / 4 points ) State your answer using the correct number of significant digits:

\[ 185 \text{ m} + (-235.5 \text{ m}) = \]

12. ( ___ / 4 points ) State your answer using the correct number of significant digits:

\[ \frac{-45 \text{ miles}}{\text{hour}} \times 3.00 \text{ hours} = \]

13. ( ___ / 10 points ) \[ \left(3a^2b^4\right)^2 \left(3ab^2z^0\right)^{-1} = \]
14. ( __ / 10 points ) \[
\frac{(2p^2q^{-3})^{-2}}{4^{-1}p^{-4}q^3} =
\]

15. ( __ / 8 points ) State your answer in both scientific notation and standard form using the correct number of significant digits:
\[
\left(3.0 \times 10^8 \text{ m/s} \right) \left(4.2 \times 10^2 \text{ sec} \right) =
\]

16. ( __ / 8 points ) State your answer in both scientific notation and standard form using the correct number of significant digits:
\[
\frac{0.000460 \text{ g}}{2.30 \times 10^{-6} \text{ L}} =
\]
17. ( ___ / 10 points ) \[
\frac{3^3 \left[ (6-2)^2 - 3 \cdot 2^2 \right] + 12}{8(8-6)} =
\]

18. ( ___ / 12 points ) A plumber earns $40 per hour, plus time and a half for overtime during the work week (more than 40 hours in one work week), plus double time for weekend work. If the plumber worked 52 hours during the work week and 7 hours on Saturday, what is the gross pay for that week? Write the starting calculation in one line using the order of operations.

19. ( ___ 10 points ) Evaluate the following expression when \( x = \frac{1}{2}, \ y = \frac{1}{3}, \) and \( z = 10: \)

\[
(2x + 3y - 6xyz)^2 =
\]

20. ( ___ 10 points ) Evaluate the following expression when \( W = 3.0 \) cm, \( L = 4.0 \) cm, and \( H = 5.0 \) cm, and state your answer using the correct number of significant digits:

\[
2(LW + WH + LH) =
\]
21. (___ 8 points) Simplify the following expression:

\[-1 + x - 2y + 3x^2 - 4y^2 + 5 - 6x + 7y - 8x^2 + 9y^2 =\]

22. (___ 8 points) Simplify the following expression:

\[2(k + 2) - 7(3k - 9) =\]

23. **Extra Credit (10 points):** If you buy 10 pens for $1.50 each, and 50 pencils for $0.10 each, and sales tax is 5.5%, write a single expression for the cost of the order, then evaluate it using the order of operations.
Chapter 2: Linear Equations and Inequalities
Section 2.1: Solving Linear Equations in One Variable

An equation is a mathematical statement involving two variable expressions and an equal sign (=). The equation makes the claim that there is some value of the variable that makes the two expressions equal at the same time. A value that makes the equation true is called a solution to the equation. To solve an equation means to find all values that make the equation true.

We will concentrate on techniques to solve a class of equations called linear equations. These are equations that have a variable but no exponents. All of the examples in this section are linear equations. The following techniques all apply to more general equation types as well, but linear equations are the ones we will be able to solve completely without introducing more advanced techniques.

Addition property of equality
The addition property of equality can be used to solve simple equations that involve only the addition or subtraction operation. Although the solution to such equations may be fairly obvious, you should make an effort to write out the steps showing how you reach the solution. This will be good practice for solving more difficult equations later on. For more complicated equations, good work habits will make everything seem easier.

Example
Solve $x - 5 = 4$

What is the solution to the equation $x - 5 = 4$? You likely said $x = 9$ without thinking too hard. How did you come up with 9? It is true that $9 - 5 = 4$ because $4 + 5 = 9$. Note that every subtraction problem has a related addition problem. We can solve the original equation by changing both sides of the equation in the same manner. In this case, we get the variable $x$ alone by adding 5 to both sides. Since $-5 + 5 = 0$, the left side of the equation only has an $x$ left:

Example
Solve $u + 19 = 7$

Solution
Add 5 to both sides to remove the $-5$ term

$u + 19 = 7$

$+5 = +5$ (add 5 to both sides)

$u = 12$

We can always check our own answers after solving an equation. After doing the work to solve, use your answer in the original equation. If you get a true statement, then you know for sure that you have the right answer. If you get a false statement, then you must have made a mistake somewhere; go back and check your work. Using $u = 12$ in the original equation, we have

$9 - 5 = 4$, which is true, so it must be the correct solution.

The addition property of equality guarantees that we can make changes (such as addition or subtraction) to an equation, provided that we treat both sides the same. If two expressions are equal, then we can add (or subtract) the same number on both sides, and the resulting expressions are still equal. We normally use the addition property of equality to create an equivalent, but simpler equation which is easier to solve than the original.

Example
Solve $u + 19 = 7$
Solution  Following the first example, we want to build an equivalent equation with the variable all alone on one side. To move the +19 term, we need to subtract 19 from both sides.

\[
\begin{align*}
u + 19 &= 7 \\
-19 &= -19 \\
u &= -12
\end{align*}
\]

In some cases, we may need to simplify expressions in the equation first. There are two simple ways to simplify: (1) Multiply through any parentheses. (2) Combine like terms on each side of the equation.

Example  Solve \(7y - 2(3y + 4) = 9\)

Solution  We need to first multiply –2 through the group \((3y + 4)\). Then the two \(y\)-terms can be combined: \(7y - 6y\) is just \(1y\).

\[
\begin{align*}
7y - 2(3y + 4) &= 9 \\
7y - 6y - 8 &= 9 \\
y - 8 &= 9
\end{align*}
\]

After simplifying, now we can use the addition property to solve the equation.

\[
\begin{align*}
y - 8 &= 9 \\
+8 &= +8 \\
y &= 17
\end{align*}
\]

**Multiplication property of equality**

Above, we found that the addition property of equality preserves the solutions when we add or subtract the same number from both sides of an equation. We can also use a similar property to solve equations where the variable term has a coefficient (a multiplier). Since the opposite of multiplication is division, we can solve equations by dividing both sides by the same number. The **multiplication property of equality** guarantees that the solutions are preserved when we change an equation by multiplying or dividing both sides by the same number.

Example  Solve equation \(5c = 40\).

Solution  Since \(5 \times 8 = 40\), we see that \(c\) must be 8. Another way to look at this is \(40 \div 5 = 8\). We can solve the equation by dividing both sides by 5.

\[
\begin{align*}
\frac{5c}{5} &= \frac{40}{5} \\
c &= 8
\end{align*}
\]

In this case, \(5c = 40\) shows the variable \(c\) multiplied by the number 5. The opposite of multiplying by 5 is dividing by 5. To keep the equation true, we must do the same thing to both sides.

Example  Solve \(3x = -30\).
Solution Here the \( x \) is multiplied by 3. The opposite operation is division by 3, so we divide both sides by 3.

\[
\frac{3x}{\cancel{3}} = \frac{-30}{\cancel{3}}\]

\( x = -10 \)

Check the answer in the original equation: –10 is the solution since \( 3(-10) = -30 \).

Example Solve \( 1.4k = 4.06 \)

Solution Divide both sides by 1.4 to remove the multiplier.

\[
\frac{1.4k}{\cancel{1.4}} = \frac{4.06}{\cancel{1.4}} \]

\( k = 2.9 \)

Check the answer in the original equation: \( k = 2.9 \) is the solution since \( 1.4(2.9) = 4.06 \).

When the coefficient is negative, we can still solve by dividing off the coefficient, including the negative sign.

Example Solve \( -9z = 47 \)

Solution Divide both sides by \( -9 \).

\[
\frac{-9z}{\cancel{-9}} = \frac{47}{\cancel{-9}}\]

\( z = -\frac{47}{9} \)

Check: \( -9 \left( \frac{-47}{9} \right) = 47 \) (the 9’s reduce out). Note that the answer may be left as an improper fraction. If the fraction does not reduce, leave the answer as is.

Solve equations that require simplification

Some equations may require simplification before you can divide. In particular, if an equation has two variable terms on the same side, combine the like terms together into a single variable term. Likewise, combine any number terms on the same side of the equation into a single number. Then we can proceed with solving the equation after the simplification is complete. Use the following procedure to step through the work needed to simplify and solve an equation.

Procedure for solving linear equations

1) First simplify each side of the equation as much as possible
a) Use the distributive property to remove parentheses groups
b) Combine like terms on each side.

2) Use the addition/subtraction properties of equality to move like terms to the same side of the equation. Separate the variable terms from the constant terms.

3) Use the multiplication property of equality to remove any remaining coefficient on the variable term by dividing both sides by the coefficient.

4) Check all solutions in the original equation to verify solutions. Correct for possible mistakes if your answers do not satisfy the equation.

Example
Solve \(3(2x + 5) = 4x + 9\)

Solution
First, distribute the multiplier to remove the parentheses.

\[6x + 15 = 4x + 9\]

Then collect like terms together on separate sides.

\[6x + 15 = 4x + 9\]
\[-4x \quad = -4x\]
\[2x + 15 = 9\]
\[-15 = -15\]
\[2x = -6\]

Finish by dividing off the coefficient of the variable term.

\[\dfrac{2x}{2} = \dfrac{-6}{2}\]
\[x = -3\]

Example
Solve \(2(3h + 4) - 9(h - 2) = 4(5h + 1)\)

Solution
First, distribute the multiplier to remove the parentheses and combine like terms on each side of the equation.

\[6h + 8 - 9h + 18 = 20h + 4\]
\[-3h + 26 = 20h + 4\]
Be very careful with the positive or negative signs on each term. The common one to miss is the double negative: \((-9)(-2) = +18\).

Then solve by moving like terms together on separate sides.

\[
-3h + 26 = 20h + 4
\]

\[
\begin{align*}
+3h &= +3h \\
26 &= 23h + 4 \\
-4 &= -4 \\
22 &= 23h
\end{align*}
\]

Remove the final multiplier using a division.

\[
\frac{22}{23} = \frac{23h}{23}
\]

\[
\frac{22}{23} = h
\]

It does not matter which side the variable ends up on, as long as it is isolated from all of the other numbers. This answer is equivalent to writing \(h = \frac{22}{23}\). Some people prefer to move the variable term to the left side every time, but the equation in either order has the same meaning.

**Your Turn!!**

Solve the following equations.

1. \(x + 5 = 9\)
2. \(x + 18 = 24\)
3. \(w + 7 = -19\)
4. \(y - 4 = -6\)
5. \(r - 10 = 12\)
6. \(x + 9 = -8\)
7. \(3y = 21\)
8. \(8x = 24\)
9. \(5f = 30\)
10. \(6c = 42\)
11. \(-3p = 21\)
12. \(4t = 14\)
13. \(-3 - 11 + k = -21 + 9\)
14. \(2w + 3w = 28 - 3\)
15. \(7x - 4x = 14 + 7\)
16. \(4z + 9 - 3z - 4 = 10\)
17. \(5x + 3 = 23\)
18. \(8x + 12 = -4\)
19. \(2k - 1 = 9\)  
20. \(8 - 5x = -7\)  
21. \(7w - 2 = 6w + 4\)  
22. \(11x = 7x + 20\)  
23. \(9y + 2 = 3y + 38\)  
24. \(8x - 3 = 4x + 17\)  
25. \(6x - 5 = 3x - 29\)  
26. \(4r + 9 - 7 = 6r + 3r - 12\)  
27. \(-7 - 4x - 5x = 9 - 6 - 5x\)  
28. \(2p + 20 + 3 = 2p + 4p - 5\)  
29. \(7(2x - 1) - 5x = x + 25\)  
30. \(9(3q + 2) - 10q = 12q - 7\)  
31. \(3y + 2(4y - 3) = 3(2y - 3)\)  
32. \(7x + 3(2x + 5) = 5(2x + 3) + 2\)  
33. \(9(y - 2) + 3(4y + 1) = 2(2y - 5)\)  
34. \(3(5b - 2) - 2(4b - 5) = 2b + 7\)

**Section 2.2: Rearranging Formulas and Solving Literal Equations**

A literal equation is an equation with more than one letter (this name comes from the fact that letters are sometimes called literals). Literal equations are usually formulas for one quantity in terms of other quantities. This section focuses on solving a literal equation for one of the “other quantities.”

**Perimeter:** Perimeter is the distance around the outside of a two-dimensional shape.  
- The perimeter of a shape with straight sides is the sum of the lengths of its sides.  
- The perimeter of a circle is called its circumference.  
- Perimeter and circumference are measured in units of length, like feet or inches or yards or meters.

**Some geometric formulas for perimeters:**

<table>
<thead>
<tr>
<th>Shape</th>
<th>Formula</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>(P = 2l + 2w)</td>
<td><img src="image" alt="Rectangle" /></td>
</tr>
</tbody>
</table>
| Circle (perimeter is called circumference) | \(C = 2\pi r\)  | ![Circle](image)  
  Pi is a numeric constant, and is approximately:  
  \(\pi \approx 3.14\)  
| Triangle                   |                  | ![Triangle](image) |
Practice:
1. What is the approximate radius of a circle that has a circumference of 22 centimeters?

2. If you have 2 feet of wood that is 2 inches wide, and you want to use it to make a square picture frame with mitered corners, what is the size of the largest picture you can frame?

3. A farmer has 800 m of fencing to enclose a rectangular field. If the width of the field is 175 m, find the length of the field.

4. Write a formula for the length of the side of an equilateral triangle in terms of the triangle’s perimeter.
**Area:** Area is the measure of how many squares of a given dimension it takes to cover a geometric shape.

- Area is measured in units of length$^2$, like feet$^2$ (i.e., square feet), or inches$^2$ (i.e., square inches), or yards$^2$ (i.e., square yards), or meters$^2$ (i.e., square meters).

### Some geometric formulas for areas:

<table>
<thead>
<tr>
<th>Geometric Shape</th>
<th>Area Formula</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>$A_{rect} = lw$</td>
<td>![Rectangle Diagram]</td>
</tr>
<tr>
<td>Circle</td>
<td>$A_{circ} = \pi r^2$</td>
<td>![Circle Diagram]</td>
</tr>
<tr>
<td>Triangle</td>
<td>$A_{tri} = \frac{1}{2}bh$</td>
<td>![Triangle Diagram]</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>$A_{trap} = \frac{1}{2}(B+b)h$</td>
<td>![Trapezoid Diagram]</td>
</tr>
<tr>
<td>Sphere</td>
<td>$A_{sphere} = 4\pi r^2$</td>
<td>![Sphere Diagram]</td>
</tr>
<tr>
<td>Box</td>
<td>$A_{box} = 2lw + 2lh + 2wh$</td>
<td>![Box Diagram]</td>
</tr>
<tr>
<td>Cylinder</td>
<td>$A_{cyl} = 2\pi rh + 2\pi r^2$</td>
<td>![Cylinder Diagram]</td>
</tr>
</tbody>
</table>
Practice:
5. The area of a triangle is 120 m². If the height is 24 m, find the length of the base.

6. The Rogers Centre (formerly the SkyDome) in Toronto, Canada has a hemispherical roof with a diameter of 630 feet. Find the area of the dome to the nearest square foot. Also, if a gallon of paint covers 400 square feet, calculate how many gallons of paint are needed to paint the Rogers Centre dome.

7. A cylinder with a radius of 12 cm has an area of 1,500 cm². Find its height.

8. Solve for the height of a cylinder in general in terms of its area and radius.

9. Solve for the height of a trapezoid in terms of its area and base lengths.
**Volume:** Volume is the measure of how many cubes of a given dimension it takes to fill a geometric shape.
- Volume is measured in units of length$^3$, like feet$^3$ (i.e., cubic feet), or inches$^3$ (i.e., cubic inches), or yards$^3$ (i.e., cubic yards), or meters$^3$ (i.e., cubic meters).

**Some geometric formulas for volumes:**

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Box</strong></td>
<td>$V_{box} = lwh$</td>
</tr>
<tr>
<td><strong>Cylinder</strong></td>
<td>$V_{cyl} = \pi r^2 h$</td>
</tr>
<tr>
<td><strong>Sphere</strong></td>
<td>$V_{sphere} = \frac{4}{3} \pi r^3$</td>
</tr>
</tbody>
</table>
| **Pyramid** | $V_{pyr} = \frac{1}{3} Bh$  
$B$ is the base area.  
The base can be any shape (not just a triangle, square, or rectangle)! |
| **Cone** | $V_{cone} = \frac{1}{3} Bh = \frac{1}{3} \pi r^2 h$ |

**Practice:**
10. A dump truck has a rectangular box that is 3 yards wide by 5 yards long. What must be the height of the box if it needs to contain a volume of 30 cubic yards?
11. If a can has a radius of 3.0 in., what must its height be if it is to hold a volume of 36 in.\(^3\)?

12. If a pyramid has a volume of 225,000 ft.\(^3\) and a square base 200 ft. on a side, what is its height?

Angle facts:
- The sum of the interior angles of a triangle is 180°.
- The sum of the interior angles of a rectangle is 360°.
- Angles that are on opposite sides of intersecting lines are equal.
- Two angles that make a straight line are called supplementary, and add up to 180°.
- Two angles that make a right angle are called complementary, and add up to 90°.

Practice:
13. If two supplementary angles are expressed as \(6x + 29°\) and \(x + 11°\), what must \(x\) be?

14. If one angle of a triangle is double the smallest angle, and the other angle is triple the smallest angle, what are the angles of the triangle?
Section 2.3: Linear Inequalities in One Variable

**Big Idea:** Inequalities are algebraic expressions related $>$, $<$, $\geq$, and $\leq$. They are used when you want the result of a calculation to be greater than or less than a certain answer. Linear inequalities are solved exactly the same as linear equalities, except that if you multiply by a negative number, you have to reverse the inequality.

\[
\begin{array}{c|c}
2 < 5 & -x < 7 \\
(-1)(2) > (-1)(5) & (-1)(-x) > (-1)(7) \\
-2 > -5 & x > -7
\end{array}
\]

To **solve an inequality** means to find all values of the variable that satisfy the inequality. Any of these values is called a **solution** to the inequality, and the set of all possible solutions is called the **solution set**.

**Properties of Inequalities:** An inequality can be transformed into an equivalent inequality by:
- adding or subtracting any quantity to both sides (the addition property of inequalities), or
- multiplying or dividing by any positive quantity (the multiplication property of inequalities).
- If both sides are multiplied or divided by a negative quantity, then the inequality symbol gets reversed.

**Steps for Solving a Linear Inequality:**
1. Simplify each side separately.
2. Isolate the variable terms on one side using the addition property of inequalities.
3. Isolate the variable using the multiplication property of inequalities.
Section 2.4: Applied Problems

Six Steps for Doing Word Problems (Applied Problems):

1. Read the problem carefully, and draw a picture if you can. Label the picture with all given information.
2. Assign a variable, and write any other unknown quantities in terms of the variable.
3. Write an equation, starting with a verbal equation if it helps.
4. Solve the equation.
5. State the answer, and verify that it makes sense.
6. Check the answer in the words of the original problem.

Number Problems

Number problems involve translating verbal forms involving only information about numbers into symbolic forms.

Example

Find a number such that 7 times a number is 3 more than the number.

Solution

Let \( x \) represent the unknown, the number.

How would you translate the words into an equation?

- 7 times a number becomes \( 7x \)
- is becomes ‘\( = \)’
- 3 more than the number becomes \( 3 + x \)

Replace the words with the translations:

\[
7x = 3 + x
\]

Solve for \( x \):

\[
7x - x = 3 + x - x
\]

\[
6x = 3
\]

\[
(6/6)x = 3/6
\]

\[
x = \frac{1}{2}
\]

Example

Find three consecutive odd numbers such that 4 times the first minus the third is the same as the second. Write an equation and solve.

Solution

Construct 3 consecutive odd numbers:

- 1st number: \( x \)
- 2nd number: \( x + 2 \)
- 3rd number: \( x + 4 \)

Note: These consecutive numbers will also work for consecutive \textit{even} numbers. Consecutive integers (not specified as odd or even) are \( x, x + 1, \) etc.

Rewrite the verbal model into an equation:

- 4 times the first becomes \( 4x \)
- minus the third becomes \( - (x + 4) \)
- is becomes ‘\( = \)’
- the second becomes \( x + 2 \)
Replace the words with the translations:

\[ 4x - (x + 4) = x + 2 \]

Solve for \( x \):

\[ 3x - 4 = x + 2 \]
\[ 3x - x - 4 = x - x + 2 \]
\[ 2x - 4 = 2 \]
\[ 2x - 4 + 4 = 2 + 4 \]
\[ 2x = 6 \]
\[ (2/2)x = 6/2 \]
\[ x = 3 \]

The first integer is 3, the second integer is 5, and the final integer is 7.

**Geometric Problems**

Geometric problems involve area, perimeter, circumference and volume.

**Example**

A rectangle is three times as long as it is wide. The perimeter is 112 cm. What are the rectangle’s dimensions?

**Solution**

The key word here is perimeter because it indicates that this is a geometric problem. The word ‘perimeter’ tells us what type of geometric problem we have. The problem also indicates that the shape is a rectangle.

To find the perimeter of the rectangle, we add up the lengths of all four sides. The formula for the perimeter is \( P = 2L + 2W \) where \( P \) is perimeter, \( L \) is length, and \( W \) is the width.

In this problem, the unknown is the dimensions of the rectangle. What do we know about the rectangle? We know that \( P \), the perimeter, is 112 cm. We also know that the length, \( L \), of the rectangle, is three times (3\( x \)) the width, \( W \).

Therefore, \[ 112 = 2(3W) + 2W \quad \text{where } L = 3W \text{ (3 times the width)} \]
\[ 112 = 6W + 2W \]
\[ 112 = 8W \]
\[ 112/8 = (8/8)W \]
\[ W = 14 \text{ cm} \]

If \( L = 3W \), then the length is 3 times 14 which is 42 cm.

**Mixture Problems:** Draw a picture with each container, a label written above each container, the amount held under each container, and the percentage held inside each container.

Mixture problems involve combining two elements in order to make a third. An example would be a store owner who has a bin of peanuts and a bin of cashews. He wants to make a third bin that contains a mixture of the peanuts and cashews together. He would use a mixture problem formula in order to determine how much of each type he needs in order to have the correct combination at the given price per pound. These problems can also be related to chemical solutions, as in the following examples.
Example
A chemical stockroom has a 20% alcohol solution and a 50% solution. How many deciliters of each should be used to obtain 90 deciliters of a 30% solution?

Solution
Let’s make a table to sort out our information. We can determine that the unknowns are how much we took from the 20% solution and the amount taken from the 50% solution.

Before we make the table, let’s look at what we have. There is some quantity taken from one mixture plus a quantity taken from a second mixture to obtain a third mixture. Therefore, our verbal equation is:

Amount of Mix A + Amount of Mix B = Amount of Mix C

<table>
<thead>
<tr>
<th></th>
<th>Mixture A</th>
<th>Mixture B</th>
<th>Mixture C</th>
</tr>
</thead>
<tbody>
<tr>
<td>% solution</td>
<td>0.20</td>
<td>0.50</td>
<td>0.30</td>
</tr>
<tr>
<td>Amt of deciliters</td>
<td>x</td>
<td>90 − x</td>
<td>90</td>
</tr>
<tr>
<td>Total Taken</td>
<td>0.20x</td>
<td>0.50(90−x)</td>
<td>0.30 * 90 = 27</td>
</tr>
</tbody>
</table>

How do we know the amounts of deciliters from A and B? We know that the total amount in the final product is 90 deciliters. Therefore, if we took (for example) 30 deciliters from A we know 60 deciliters were taken from B because the total amount taken is 90 deciliters. Therefore, we let x represent the amount taken from A and the difference between what the end product contains and what we have taken from A is what was taken from B.

Now, substitute the table values into the verbal equation.

\[
0.20x + 0.50(90-x) = 27
\]
\[
0.20x + 45 – 0.50x = 27
\]
\[
-0.30x = -18
\]
\[
x = 60
\]

60 deciliters were taken from the 20% solution and 30 deciliters were taken from the 50% solution.

Example
How many grams of 10% tin solder must be added to 800 g of 15% tin solder in order to make a final solder which is 12% tin?

Solution
Let x stand for the amount of 10% tin solder. When we combine the x grams of the 10% tin solder with the 800 g of 15% solder, the final 12% tin solder must contain 800 + x grams. The following table is set up in a different manner than the first. You might find this table serves you as a more useful tool in putting the information together.

<table>
<thead>
<tr>
<th>10% Tin</th>
<th>+</th>
<th>15% Tin</th>
<th>=</th>
<th>12% Tin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount of Solder</td>
<td>x</td>
<td>800 g</td>
<td>x+800 g</td>
<td></td>
</tr>
<tr>
<td>% Tin</td>
<td>10%</td>
<td>15%</td>
<td>12%</td>
<td></td>
</tr>
<tr>
<td>Amount of Tin</td>
<td>0.10x</td>
<td>0.15(800) = 120</td>
<td>0.12(x+800)</td>
<td></td>
</tr>
<tr>
<td>0.10x</td>
<td>+</td>
<td>120</td>
<td>=</td>
<td>0.12(x+800)</td>
</tr>
</tbody>
</table>
The tin in the final 12% solder must all have come either from the 10% tin solder or the 15% tin solder. This translates into the equation, 

\[ 0.10x + 120 = 0.12(800 + x) \]

which we solve for \( x \).

\[
\begin{align*}
0.10x + 120 &= 0.12(800 + x) \\
0.10x + 120 &= 0.12(800) + 0.12x \\
0.10x + 120 &= 96 + 0.12x \\
0.10x - 0.10x + 120 &= 96 + 0.12x - 0.10x \\
120 &= 96 + (0.12 - 0.10)x \\
\text{or} & \\
96 + 0.02x &= 120 \\
-96 + 96 + 0.02x &= -96 + 120 \\
0.02x &= 24 \\
\frac{0.02x}{0.02} &= \frac{24}{0.02} \\
x &= 1200
\end{align*}
\]

This answer can be verified by plugging it into the previous table, where indeed it is true that 120 + 120 = 0.12(2000).

**Interest Problems:** Interest = Principal × Interest Rate \( \Rightarrow I = pr \).

**Money Problems:** number \( \times \) value of one item = total value.

**Speed Problems:** distance = rate \( \times \) time \( \Rightarrow d = rt \). Draw a picture with all distances, rates (speeds), and times labeled. The most common distance formula is distance = rate \( \times \) time.

**Example** A car leaves a city traveling at 60 mph per hour. How long will it take a second car traveling at 80 mph per hour to catch up to the first car if it leaves 2 hours later?

**Solution** Let’s make a table. The table will compare the distance, rate, and time for both cars. Notice that the distance both cars travel is the same. Our verbal equation will look like:

**Distance Car 1 Travels = Distance Car 2 Travels**

<table>
<thead>
<tr>
<th></th>
<th>Rate</th>
<th>Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car 1</td>
<td>60 mph</td>
<td>( x )</td>
<td>60x</td>
</tr>
<tr>
<td>Car 2</td>
<td>80 mph</td>
<td>( x - 2 )</td>
<td>80(x – 2)</td>
</tr>
</tbody>
</table>

Car 2 has time \( x - 2 \) because it will take 2 hours LESS for the second car to reach the destination. Imagine you leave at 9 a.m. for Kansas City. You arrive at 1:00. Your friend leaves at 11:00 and still arrives in Kansas City at 1:00. You both arrived at the same time, but it took your friend 2 hours LESS THAN you to get there.
Remember the verbal equation for this problem. Now that I have the distance for both cars, I can substitute my unknowns into the formula.

\[60x = 80(x - 2)\]
\[60x = 80x - 160\]
\[-20x = -160\]
\[x = 8\]

It will take the first car 8 hours and the second car 6 hours to catch up with each other.

1. The sum of two consecutive odd integers is 444. What are the two numbers?

2. A rectangle is three times as long as it is wide. Its perimeter is 48 in. What are the rectangle’s dimensions?

3. It takes you four hours to drive to a city. If you had driven 15 mph faster, you would have gotten there in only three hours. How far did you drive?

4. John’s age is one year less than twice Mary’s age. Their combined age is 74 years. How old are they?

5. In Sarah’s pocket there is $2.51 in change. There is one more dime than quarters, but one less dime than twice the number of nickels. The number of pennies is three less than twice the number of dimes. How many of each type of coin are there?

6. How many grams of 16% tin solder must be added to 200 g of 25% tin solder to make a final solder which is 20% tin?

**Section 2.5: Percent Problems**

Percent problems usually involve discount problems, sales tax problems, or grading problems. Percent means ‘per 100’. Percents are also interpreted as a part of the whole. If we have 35%, then that means we have 35 parts ‘per 100’ or 35% of the whole. They are another way to write fractions and decimals.

Converting from percent to decimal:

\[28\% \text{ means } 28 \text{ per } 100\]

As a fraction, it is written \[\frac{28}{100}\].

As a decimal, we move the decimal point two places to the left:

\[28\% = 28 \text{ and, as a decimal, becomes } 0.28\]
\[12.5\% \text{ is } 0.125 \text{ as a decimal}\]
\[0.62\% \text{ is } 0.0062 \text{ as a decimal}\]

Notice that it is easier to convert from a percent to a decimal and then to a fraction because the number of places after the decimal indicates how many zeros in the denominator when converting from decimal to fraction.

\[0.0062 = \frac{62}{10000}\]
\[0.125 = \frac{125}{1000}\]
\[0.28 = \frac{28}{100}\]
If you are given the decimal and must convert to a percentage, move the decimal two places to the right:

\[ 0.75 = 75\% \quad 8.27 = 827\% \quad 0.057 = 5.7\% \]

When solving word problems involving percents, it is necessary to convert the percentages to decimals before doing any calculations.

**Example**

A store is giving 30% off the ticketed price. If a shirt has a ticketed price of $42, how much is the new price after the discount?

**Solution**

To solve this, we would use the formula:

\[
\text{New Price} = \text{Old price} - (\% \text{ discount} \times \text{Old Price})
\]

\[
\text{New Price} = \text{Old Price} \times (1 - \% \text{ discount})
\]

\[
= 42 \times 0.70
\]

\[
= $29.40
\]

Note that when you are buying this item on sale, you are paying 70\% of the original price. The store is taking off 30\% of the original price.

We could also write this as

\[
\frac{70}{100} = \frac{x}{42}
\]

where \(x\) represents the part you are paying of the whole.

Using this type of equation is called **proportions**. A proportion is two ratios set equal to each other. It says that 70 is to 100 as what part is to 42? To solve proportions, we use cross multiplication.

\[
\frac{9}{x} = \frac{5}{6}
\]

\[
9 \times 6 = 5x
\]

\[
x = \frac{54}{5}
\]

\[
x = 10.8
\]

For percent problems, we can use proportions

\[
\frac{\% \text{ part}}{100 \text{ whole}} = \frac{\text{part}}{\text{whole}}
\]

For each problem, we must determine which is missing: percent, part, or whole. Use the proportion to fill in what we know and use cross multiplication to solve for the unknown.

**Example**

27 is 54\% of what value?

**Solution**

Percent: \(54\)

Part: \(27\)

Notice that 27 is only 54\% of some larger number.

Whole: \(\text{Unknown, } x\)
\[
\frac{\%}{100} = \frac{\text{part}}{\text{whole}}
\]
\[
\frac{54}{100} = \frac{27}{x}
\]
\[
54x = 27 \times 100
\]
\[
54x = 2700
\]
\[
\frac{54x}{54} = \frac{2700}{54}
\]
\[
x = 50
\]

**Alternative Solution**

Use the key words/phrases above to translate this percentage sentence into an equation:

- is becomes ‘=’
- of becomes \(x\) (multiplication)
- ‘what value’ is your unknown-replace with a variable

\[
27 = 0.54 \times x
\]

Divide both sides by 0.54 to get \(x\) by itself:
\[
x = 50
\]

Obviously, this process is much simpler. However, in some word problems, you might still want the proportion.

**Example**

What percent of $160 is $200?

**Solution**

‘what percent’ is your unknown, \(x\)

of means times/multiply

is means =

\[
x \times 160 = 200
\]

\[
160x = 200
\]

\[
\frac{160x}{160} = \frac{200}{160}
\]

\[
x = 1.25 = 125\%
\]

**Example**

125% of what value is $423?

**Solution**

of means times/multiply

‘what value’ is your unknown, \(x\)

is means =

\[
1.25 \times x = 423
\]

\[
1.25x = 423
\]

\[
\frac{1.25x}{1.25} = \frac{423}{1.25}
\]

\[
x = \$338.40
\]
A common percent application is sales tax. To determine the amount of sales tax, we multiply the sales tax times the price of the item. Check out this quick way to determine the price at checkout.

\[
\text{Price of Item} = $35 \quad \text{Sales Tax} = 6.5\%
\]
\[
\text{Checkout Price} = \text{Price of Item} + (\text{Sales Tax} \times \text{Price of Item})
\]
\[
= \text{Price of Item}(1 + \text{Sales Tax})
\]
\[
\text{Checkout Price} = 35(1 + 0.065)
\]
\[
= $37.28
\]

**Example**
The checkout price of a new TV was $258.48. If the sales tax is 5.5%, what was the sales price?

**Solution**
\[
\text{Checkout Price} = \text{Price of Item}(1 + \text{Sales Tax})
\]
\[
258.48 = P(1 + 0.055)
\]
\[
258.48 = P(1.055)
\]
\[
\text{Divide both sides by 1.055 to isolate } P.
\]
\[
P = $245.00
\]

**Example**
A DVD player regularly priced at $199.99 is discounted 20%. What is the new selling price?

**Solution**
\[
\text{Selling Price} = \text{Original Price} - (\text{Original Price} \times \text{Discount Amt})
\]
\[
\text{Selling Price} = \text{Original Price} (1 - \text{Discount Amt})
\]
\[
\text{Selling Price} = 199.99(0.80)
\]
\[
= $160.00
\]

**Example**
A sales person earns $250 per week plus a 2.5% commission. If the person desires a week’s gross pay of $900, how much merchandise must be sold?

**Solution**
\[
\text{Gross Pay} = 250 + \text{Commission Rate} \times \text{Amt of Merchandise Sold}
\]
\[
900 = 250 + (0.025 \times A)
\]
\[
900 = 250 + 0.025A
\]
\[
900 - 250 = 250 - 250 + 0.025A
\]
\[
650 = 0.025A
\]
\[
\frac{650}{0.025} = \frac{0.025A}{0.025}
\]
\[
A = $26,000
\]

The person must sell $26,000 in merchandise to make $900 for the week.
We can also percent problems to solve for Percent Tolerance Percent Error. When a measurement is specified on a precision part, such as a diameter of 0.150 in plus or minus 0.001 in, the 0.001 in is called the tolerance. This means that any measured diameter between 0.149 in and 0.151 in is acceptable. Another way of specifying the tolerance is the percent tolerance defined by the following formula:

\[
\text{Percent Tolerance} = \left(\frac{\text{tolerance}}{\text{specified value}}\right) \times 100\%
\]

Again multiplying by 100% changes the fractional tolerance into a percent tolerance. The following examples illustrate percent tolerance.

**Example** A resistor is rated at $85\Omega \pm 2\Omega$. What is the percent tolerance?

**Solution**

\[
\text{Percent Tolerance} = \frac{2\Omega}{85\Omega} \times 100\% = 2.35\%
\]

**Example** An acceptable voltage reading is 15.0 V with a percent tolerance of 3%. Express this tolerance in volts.

**Solution**

\[
\text{Tolerance} = 3\% \times 15.0 \text{V} = 0.03 \times 15.0\text{V} = 0.45\text{V}
\]

So the voltage is 15.0 V ± 0.45 V.

Percent error is similar to percent tolerance. When an actual measurement is made and compared to the “true” or “specified” value, the percent error is defined by the following formula:

\[
\text{Percent Error} = \left(\frac{\text{measured value} - \text{specified value}}{\text{specified value}}\right) \times 100\%
\]

Usually if the calculated percent error is negative the minus sign is ignored. This is because we are concerned with “how close” we are to the specified value and not whether we are above or below it. The following illustrates a percent error calculation.

**Example** A part is specified as having a diameter of 0.250 in. The manufactured part measures 0.254 in. What is the percent error?

**Solution**

\[
\text{Percent Error} = \frac{(0.254 \text{ in} - 0.250 \text{ in})}{0.250 \text{ in}} \times 100\% = 1.60\%
\]

**Your Turn!!**

Write each of the following numbers as a percent:

1. 0.37
2. 0.012
Write each percent as a decimal number:

6. 22.5%  6) ________________
7. 211%  7) ________________

Write each percent as a fraction reduced to lowest terms:

8. 125%  8) ________________
9. $\frac{5}{6} \%$  9) ________________

Solve for the following unknowns:

10. What is 16.5% of 128?  10) ________________
11. 125% of $950$ is what amount?  11) ________________
12. 53 is what percent of 120?  12) ________________
13. 12 is to 80 like 45 is to what number?  13) ________________
14. 12 is to 60 like what amount is to 90?  14) ________________
15. $125$ is 36% of what amount? (Round to the nearest penny.)  15) ________________
16. What percent of $\frac{3}{8}$ is $\frac{5}{16}$?

16) ________________

17. A license cost $120. If the cost increases 7.5%, what is the new cost of this license?

17) ________________

18. If sales tax is 5.5%, what would be the check out price of a band saw with a list price of $289?

18) ________________

19. From a 15.0 lb cylinder 1.6 lb of material is removed during machining. What percent of material was removed?

19) ________________

20. An electrical resistor is rated at 85 ohms plus or minus 5%. Express this tolerance in ohms.

20) ________________

21. A piston is to have a diameter of 0.787 in ± 0.003 in. What is the percent tolerance?

21) ________________

22. Specifications call for a pin to be 1.500 in long. If the finished pin measures 1.504 in., what is the percent error?

22) ________________

23. If an electric drill usually selling for $89.95 is on sale at a discount of 25%, what is the new list price?

23) ________________
24. A new car is advertised as selling for $14,220. This price reflects a 9% discount. What was the original (list) price? (Round to the nearest penny.)

25. An assembly line is shut down for inspection if the fraction of defective products exceeds 0.5%. If the normal day’s production is 10,500 units, at most how many defective units can there be if the line is not to be shut down?

26. A salesperson is paid $380 per week plus a 2.2% commission. What is the person’s sale total if the gross pay for a given week is $1,395? (Round to the nearest penny.)

Section 2.6: Percent Problems from Finance

Percent problems also include problems dealing with interest earned, monthly payments on a loan, or number of years to pay off a loan. It is not required that students memorize these formulas, but you should understand how to use them appropriately.

When getting a loan from a lender, we must pay back both the amount borrowed (the principal) plus interest for the temporary use of the lender’s money. In a simple interest loan everything is paid in one lump sum. For example, if you borrow $1500 at an interest of 5.0%, then you must pay back 100% of the principal plus 5.0% of the principal or (1.05)($1500) = $1575.

Most conventional loans are paid back in a sequence of monthly payments. In each payment a portion is used to pay the interest owed on the remaining debt and what is left over is used to reduce the debt, i.e., is paid against principal. A conventional loan is characterized by the following four parameters.

1. The initial principal symbolized by P.
2. The annual percentage rate (APR) of interest symbolized by R.
3. The number of years over which the loan is paid off (the period of amortization) symbolized by N.
4. The monthly payment symbolized by M.
To calculate \( M \) knowing \( P, R, \) and \( N \) use the formula:

\[
M = \frac{PR}{12 \cdot \left[ 1 - \frac{1}{(1 + \frac{R}{12})^{12N}} \right]}
\]

To calculate \( N \) knowing \( P, R \) and \( M \) use the formula:

\[
N = \frac{-12 \cdot \log \left( 1 - \frac{R \cdot P}{12 \cdot M} \right)}{-12 \cdot \log \left( 1 + \frac{R}{12} \right)}
\]

To calculate \( P \) knowing \( M, R, \) and \( N \) use the formula:

\[
P = \frac{12 \cdot M}{R} \cdot \left[ 1 - \frac{1}{(1 + \frac{R}{12})^{12N}} \right]
\]

In the formula for \( N \), log is the logarithm function, which is found on any scientific calculator. These formulas are rather complicated although they can be easily computed using a spreadsheet program such as Excel or Quatro Pro. With patience they can also be done on a scientific calculator. To illustrate these calculations consider the following three examples.

**Example**

If you borrow $12,000 at an annual percentage rate of 2.3% to be paid off over 4 years, what is your monthly payment?

**Solution**

\( P = \$12,000, \ R = 0.023 \) and \( N = 4 \), so we need to use the formula for \( M \).

It is easier to fill in the values and then take the problem step by step to solve. If you have a graphing calculator, these formulas can be entered all at once as long as you watch the parentheses.

\[
M = \frac{PR}{12 \cdot [1 - 1 \div (1 + R/12)^{12N}]} \\
M = \frac{12000 \cdot 0.023}{12 \cdot [1 - 1 \div (1 + 0.023/12)^{12(4)}]} \\
M = \$261.92
\]

We can also percent problems to solve for Percent Tolerance Percent Error. When a measurement is specified on a precision part, such as a diameter of 0.150 in plus or minus 0.001 in, the 0.001 in is called the **tolerance**. This means that any measured diameter between 0.149 in and 0.151 in is acceptable. Another way of specifying the tolerance is the percent tolerance defined by the following formula:
Percent Tolerance = \frac{\text{tolerance}}{\text{specified value}} \times 100\%

Again multiplying by 100% changes the fractional tolerance into a percent tolerance. The following examples illustrate percent tolerance.

**Example**  
A resistor is rated at $85\Omega \pm 2\Omega$. What is the percent tolerance?

**Solution**  
Percent Tolerance = \frac{2\Omega}{85\Omega} \times 100\% = 2.35\%

**Example**  
An acceptable voltage reading is 15.0 V with a percent tolerance of 3%. Express this tolerance in volts.

**Solution**  
Tolerance = 3\% \times 15.0 \text{ V} = 0.03 \times 15.0 \text{ V} = 0.45 \text{ V}

So the voltage is 15.0 V ± 0.45 V.

Percent error is similar to percent tolerance. When an actual measurement is made and compared to the “true” or “specified” value, the percent error is defined by the following formula:

\[
\text{Percent Error} = \frac{\text{measured value} - \text{specified value}}{\text{specified value}} \times 100\%
\]

Usually if the calculated percent error is negative the minus sign is ignored. This is because we are concerned with “how close” we are to the specified value and not whether we are above or below it. The following illustrates a percent error calculation.

**Example**  
A part is specified as having a diameter of 0.250 in. The manufactured part measures 0.254 in. What is the percent error?

**Solution**  
Percent Error = \frac{(0.254 \text{ in} - 0.250 \text{ in})}{0.250 \text{ in}} \times 100\% = 1.60\%

**Your Turn!!**

1. What is the monthly payment required to pay off a $12,000 loan in two years at an annual percentage rate of 2.7\%? (Round to the nearest penny.)

   1) ________________

2. What is the largest amount which can be borrowed over three years at 4.5\% APR if the largest affordable monthly payment is $279? (Round to the nearest ten dollars.)
2) ________________

3. How long would it take to pay off $15,000 at 5.2% APR if the monthly payment is $450? (Round to the nearest tenth of a year.)

3) ________________

4. You can afford 15% of your monthly income of $2300 on car payments. If the quoted annual percentage rate of the loan is 2.5% over three years, what is the most you can borrow? (Round to the nearest ten dollars.)

45) ________________

Section 2.7: Direct and Inverse Variation Problems
If a ratio equals a constant, the two variables vary directly or are directly proportional. When we say, ‘y varies directly with x,’ or ‘y is directly proportional to x,’ then the linear model is

\[ y = kx \]

where \( k \) is the constant of proportionality or rate of change.

If you think about the distance formula (\( d = rt \)), notice that \( r \), rate, is a constant of proportionality. The dependent variable is \( d \), distance, because it depends upon the time, \( t \), which you drive. However, your rate of speed, \( r \), stays constant. The formula is seen as, ‘distance varies directly with time.’ Another example is your hourly wages. Let’s use the equation \( s = wh \) where \( s \) is the salary, \( w \) is the hourly wage, and \( h \) is the number of hours worked. Again, \( s \) is the dependent variable because your salary depends upon the number of hours, \( h \), that you work. And unless you somehow pull a raise out of your boss, your hourly wage, \( w \), remains constant. So, ‘salary is directly proportional to the number of hours worked.’

**Example**

\( P \) varies jointly with \( r \) and \( s \). If \( P = 16 \) when \( r = 5 \) and \( s = -8 \), find \( P \) when \( r = 2 \) and \( s = 10 \).

**Solution**

Let \( k \) represent the constant of proportionality. The representation of the relationship above is \( P = krs \). The information given does not provide the value for \( k \), but we can find it.

\[
\begin{align*}
P &= krs \\
16 &= k(5)(-8) \\
16 &= -40k \\
k &= 16/(-40) = 2/(-5) = -0.4
\end{align*}
\]

Now we can complete the equation: \( P = -0.4rs \)
And, the final part of the problem:

\[
P = -0.4rs \\
P = -0.4(2)(10) \\
P = -8
\]

There are other types of variations we can work with. \( y \) can also be inversely proportional to \( x \). This is represented by \( y = \frac{k}{x} \).

**Example**  
R is inversely proportional to the square of \( I \). If \( I = 25 \), then \( R = 100 \). Find the constant of proportionality.

**Solution**  

\[
R = \frac{k}{I^2} 
\]

Equation

\[
100 = \frac{k}{25^2} 
625 \times 100 = k 
\]

\[
k = 62,500
\]

Often a quantity depends on more than one variable. For example, the pressure of a gas depends on the amount of gas present (the number of “moles”, \( n \)), the absolute temperature (\( T \)), and the volume occupied by the gas (\( V \)). The relationship between these variables is that the pressure of a gas varies as the product of the amount of gas and the absolute temperature and inversely as the volume occupied by the gas. The phrase, “varies as the product”, means the same thing as varies directly as the product. Thus, the relationship is described by the following formula.

\[
P = \frac{RnT}{V}
\]

Here, by standard usage, the proportionality constant is called \( R \).

Problems involving variation are often of the form of providing two different sets of data with an unknown quantity in one of the data sets. The simplest way to solve this kind of problem is to convert it into a proportion as the following examples illustrate.

**Example**  
A drive gear having 80 teeth and rotating at 40 rpm meshes with a second gear having 16 teeth. How fast does the second gear rotate?

**Solution**  
This is a proportion problem since for two meshing gears the number teeth per minute that pass the contact point must be the same for both gears. Let the number of teeth be \( N \) and the rate of rotation (measured in \( \text{rpm} \)) be \( r \), then \( N \cdot r = k \) (\( k \) a constant). Stated differently, \( \frac{Nr}{N} = r = \frac{k}{N} \).

This means that the rate of rotation for meshing gears is inversely proportional to the number of teeth. The fewer teeth, the faster the gear rotates. Let \( N_1 = 80, r_1 = 40 \text{ rpm} \) and \( N_2 = 16 \). The use of subscripts enables us to label the values that belong together in the same data set. The
unknown quantity we seek is $C$, the rotation rate of the second gear. Since $r_1 = \frac{k}{N_1}$ and $r_2 = \frac{k}{N_2}$, by taking the ratio of $r_2$ to $r_1$ the constant $k$ drops out.

$$\frac{r_2}{r_1} = \frac{k}{N_2} \div \frac{k}{N_1} = \frac{k}{N_2} \cdot \frac{N_1}{k} = \frac{N_1}{N_2}$$

Since $r$ is inversely proportional to $N$, the subscripts on $N$ are opposite (the numerator and the denominator are switched) to those on $r$. Plugging in the values for the variables gives the proportion, $\frac{r_2}{40 \text{ rpm}} = \frac{80}{16}$, which we solve for $r_2$.

$$\frac{r_2}{40 \text{ rpm}} = \frac{80}{16}$$
$$16r_2 = 40 \text{ rpm} \cdot 80$$
$$\frac{16r_2}{16} = \frac{40 \text{ rpm} \cdot 80}{16} = 40 \text{ rpm} \left( \frac{80}{16} \right)$$
$$r_2 = 40 \text{ rpm} \cdot 5 = 200 \text{ rpm}$$

**Your Turn!!**

1. A drive gear having 100 teeth and rotating at 40 rpm meshes with a second gear having 20 teeth. How fast does the second gear rotate?

2. If $y$ is directly proportional to $x$ and $x = 10$ when $y = 4$, what is $x$ when $y = 12$?

3. If $x$ is inversely proportional to $y$ and $x = 24$ when $y = 3$, what is $x$ when $y = 6$?

4. If $y$ varies directly as the product of $x^2$ and $t$, what is $t$ when $y = 10$ and $x = 5$, if $y$ is 15 when $x = 3$ and $t = 5$?

For the problems 5 - 7, use the information that the pressure of a gas, $P$, varies directly as the absolute temperature $T$ and inversely as the volume, $V$.

5. If $P$ is 10.5 psi when $T$ is 400 K and $V$ is 3.5 L, what is $P$ when $T$ is 380 K and $V$ is 3.0 L?

6. If $P$ is 25.0 psi when $T$ is 400 K and $V$ is 6.00 cubic feet, what is $T$ when $P$ is 20.0 psi and $V$ is 9.00 cubic feet?

7. If $P$ is 1.0 atmosphere when $T$ is 300 K and $V$ is 22.0 L, what is $V$ when $T$ is 450 K and $P$ is 3.2 atmospheres?

8. The resistance, $R$, of a wire varies directly as the length, $l$, and inversely as the diameter, $d$, squared. If $R$ is 3.64 ohms when $l = 5.0$ m and $d = 1.25$ mm, what is the resistance of a wire made of the same material that is 20.0 m long and has a diameter of 2.50 mm?
Chapter 2 Practice Exam

1. Find percent notation for \( \frac{7}{8} \).

2. Find fractional notation for 9.75%.

3. Find the decimal notation for 20%.

4. Solve: 175 is what percent of 100?

5. Solve: 80 is what percent of 40?

6. Solve: What is 55% of 262?

7. The sales tax rate in a city is 8.2%. Find
   a) the tax charged on a purchase of $238
   b) the total cost

8. Of all the people that attend movies, 67% are in the 12-29 age group. At one theater, 600 people attended a showing of a certain movie. How many were in the 12-29 age group?

9. Find the missing values:

<table>
<thead>
<tr>
<th>Marked Price</th>
<th>Rate of Discount</th>
<th>Discount</th>
<th>Sale Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>$109.00</td>
<td></td>
<td>$21.80</td>
<td></td>
</tr>
</tbody>
</table>

10. A ‘40% off sale’ begins today at Wanda’s Women’s Wear. What is the sale price of women’s wool coats normally priced at $250?

11. The population of a state was 1,563,000 with 44.5% represented by a minority. What is the best estimate of the minority population in this state?
12. Joe took an exam with 16 questions. He got 3 of the questions wrong. What percent of the questions did he get right?

13. A rectangle is three times as long as it is wide. Its perimeter is 24 in. What are the rectangle's dimensions? Note: The perimeter is the sum of all four sides of a rectangle.

14. A house cleaning service charges a fee of $50 a visit and then $10.75 per hour for cleaning a home. The Robert's had the service clean their house last spring and it cost them $211.25. How long did the cleaning service take to clean the Robert's house?

15. Determine \( n \):

\[
\frac{12}{n} = \frac{9}{16}
\]

16. Most mammals breathe about once for every 4 heartbeats. If a runner after finishing a race has a heart rate of 125 beats per minute, how many breaths is the runner taking in one minute?

17. The time it takes to complete a task varies inversely with the number of people performing the task. If Jim's lawn service using two mowers, they can cut a large lawn in 5.5 hours, how long would it take to cut the grass on the same lawn if 7 mowers were used?

18. If \( y \) varies jointly as \( x \) and the square of \( z \), and if \( y = 192 \) when \( x = 12 \) and \( z = 9 \), then what is \( y \) when \( x = 2 \) and \( z = 16 \)?

19. The resistance, \( R \), of a wire varies directly as the length, \( l \), and inversely as the diameter, \( d \), squared. If \( R \) is 1.50 ohms when \( l = 2.0 \) m and \( d = 4.0 \) mm, what is the resistance of a wire made of the same material that is 10.0 m long and has a diameter of 5.0 mm?

20. It takes you three hours to drive to a city. If you had driven 10 mph slower, it would have taken you 36 minutes longer. How far did you drive?
Chapter 3: Algebra and the Graph of a Line

Section 3.1: Graphing a Linear Equation Using a Table of Values

**Big Skill:** You should be able to graph linear algebraic equations by creating a table of values and then plotting the points and connecting the dots to form a straight line. You should also be able to read a given graph for approximate values.

When there is a relationship between two quantities, it is helpful to understand that relationship with a picture. The main way we picture mathematical relationships is with a graph.

A graph is a picture formed by dividing the plane into four regions, called quadrants, with a pair of number lines that intersect at right angles. We put a point on the graph to represent the relationship for a single pair of values by moving horizontally along the first number line, and then vertically parallel to the second number line. The collection of all such points for the relationship is called the graph of the relationship. When we graph “A vs. B”, the vertical axis represents the amount of A, and the horizontal axis represents the amount of B. B is called the independent variable, and A is called the dependent variable.

**Technique #1 for graphing a line: USING A TABLE OF VALUES**
1. Solve the linear equation for the dependent variable (y if y and x are the only two variables in the equation.)
2. Pick some random values of the independent variable (x if y and x are the only two variables in the equation).
3. Calculate the values for the dependent variable using the equation; make a table of the values.
4. Graph the points and connect them with a straight line.

**Your Turn!!**

1) Graph the following twelve linear equations.

*In Problems 25–52, graph each equation by plotting points.*

<table>
<thead>
<tr>
<th>Equation</th>
<th>Equation</th>
<th>Equation</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 4x$</td>
<td>$y = 2x$</td>
<td>$y = \frac{-1}{2}x$</td>
<td>$y = \frac{-1}{3}x$</td>
</tr>
<tr>
<td>$y = x + 3$</td>
<td>$y = x - 2$</td>
<td>$y = -3x + 1$</td>
<td>$y = -4x + 2$</td>
</tr>
<tr>
<td>$y = \frac{1}{2}x - 4$</td>
<td>$y = \frac{-1}{2}x + 2$</td>
<td>$2x + y = 7$</td>
<td>$3x + y = 9$</td>
</tr>
</tbody>
</table>
Section 3.2: Graphing a Linear Equation Using the Slope Intercept Method

**Big Idea:** A shortcut for graphing lines is to understand that when you solve for the dependent variable, then the number multiplying the constant is where the line crosses the vertical axis, and the number multiplying the dependent variable is the “slope,” which tells you how far up and over to move to get to the next point.

One thing to notice when making a table of values to graph a line is that if you pick your “random” values of the independent variable to change by the same amount every time, then the dependent variable also changes by the same amount every time. Here is an example from Section 3.1 that illustrates this:

**Practice:**
3. A cell phone package has a flat fee of $10 a month, plus 10 cents a minute for every minute you talk. Write an equation for the cost of this package, and then graph it.

<table>
<thead>
<tr>
<th>Equation Solved for the Dependent Variable</th>
<th>Table of Values</th>
<th>Graph of Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( C = # \text{ minutes} )</td>
<td>Independent</td>
<td>Dependent</td>
</tr>
<tr>
<td>( \text{(independent = #) Free to choose} )</td>
<td>( M )</td>
<td>( C )</td>
</tr>
<tr>
<td>Let ( C = \text{cost} )</td>
<td>0 min.</td>
<td>$10</td>
</tr>
<tr>
<td>( \text{(dependent \Rightarrow depends on minutes)} )</td>
<td>100 min.</td>
<td>$20</td>
</tr>
<tr>
<td>( \text{cost = $10 plus 10¢ for each minute} )</td>
<td>200 min.</td>
<td>$30</td>
</tr>
<tr>
<td>( C = $10 + \left( \frac{0.10}{\text{min.}} \right) # )</td>
<td>500 min.</td>
<td>$50</td>
</tr>
<tr>
<td></td>
<td>750 min.</td>
<td>$75</td>
</tr>
<tr>
<td></td>
<td>1000 min.</td>
<td>$100</td>
</tr>
</tbody>
</table>

This constant rate of change for both variables is captured in a ratio called the “slope” of the line. The slope of a line is how much the dependent variable changes divide by how much the dependent variable changes. For the example above:

\[
\text{slope} = \frac{\text{change in cost}}{\text{change in minutes}} = \frac{$10}{100 \text{ minutes}} = $0.10 \text{ per minute}
\]

Notice that the slope is just the rate from the original statement of the problem, and also that it is the number multiplying \( m \), the independent variable. Simply put, the slope converts a change in minutes to a change in dollars in this example.
A second thing to notice from the example above is that when \( m = 0 \) minutes, the cost was \( c = $10 \), and that $10 was the constant in the equation we got when we solved for the dependent variable:

\[
c = \frac{$.10}{1 \text{ minute}} \cdot m + [10]
\]

The point on the graph (0 minutes, $10) is called the “vertical intercept,” or more commonly the “y intercept,” because most math books are so boring that they only ask you to graph \( x \) and \( y \) as the independent and dependent variables, respectively.

**Technique #2 for graphing a line: SLOPE INTERCEPT METHOD**

- Solve the linear equation for the dependent variable (i.e., solve for \( y \) if \( x \) and \( y \) are the only two variables in the equation)
- The constant term is the vertical intercept, or the \( y \)-intercept. Plot it.
- The factor of the \( x \) term is the slope. Use it to count up and over (or down and backward) to plot more points on the line.

Your Turn!!

Use the slope intercept method when graphing!

Graph \(-6x + 5y = -10\)

Graph \(5x - 8y = 64\)
<table>
<thead>
<tr>
<th>Graph 10 $10x - 5y = 30$</th>
<th>Graph 8 $8x - 6y = -54$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph 10" /></td>
<td><img src="image2.png" alt="Graph 8" /></td>
</tr>
<tr>
<td>Graph -3 $-3x + 7y = -56$</td>
<td>Graph 5 $5x - 7y = -56$</td>
</tr>
<tr>
<td><img src="image3.png" alt="Graph -3" /></td>
<td><img src="image4.png" alt="Graph 5" /></td>
</tr>
<tr>
<td>Graph -$x + 9y = -81$</td>
<td>Graph 6 $6x + 4y = 28$</td>
</tr>
<tr>
<td><img src="image5.png" alt="Graph -$x + 9y = -81$" /></td>
<td><img src="image6.png" alt="Graph 6" /></td>
</tr>
</tbody>
</table>
Graph $-x - 3y = 15$

Graph $-2x + 2y = 16$

Graph $-8x + 5y = -45$

Graph $-10x + 6y = -24$

Graph $-8x - 9y = 63$

Graph $5x + 2y = 16$
<table>
<thead>
<tr>
<th>Equation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7x - 5y = 30$</td>
<td><img src="image" alt="Graph 7x - 5y = 30" /></td>
</tr>
<tr>
<td>$-6x + 3y = -6$</td>
<td><img src="image" alt="Graph -6x + 3y = -6" /></td>
</tr>
<tr>
<td>$6x + 2y = 10$</td>
<td><img src="image" alt="Graph 6x + 2y = 10" /></td>
</tr>
<tr>
<td>$3x - 3y = -21$</td>
<td><img src="image" alt="Graph 3x - 3y = -21" /></td>
</tr>
<tr>
<td>$4x - 5y = 35$</td>
<td><img src="image" alt="Graph 4x - 5y = 35" /></td>
</tr>
<tr>
<td>$6x - 5y = 25$</td>
<td><img src="image" alt="Graph 6x - 5y = 25" /></td>
</tr>
</tbody>
</table>
Section 3.3: Graphing a Linear Equation Using Intercepts

Technique #3 for graphing a line: INTERCEPT METHOD

- Set the independent variable to zero, then solve the linear equation for the dependent variable (i.e., set $x = 0$ and solve for $y$ if $x$ and $y$ are the only two variables in the equation).
- Plot the point $(0, y)$. This is called the $y$-intercept.
- Set the dependent variable to zero, then solve the linear equation for the independent variable (i.e., set $y = 0$ and solve for $x$ if $x$ and $y$ are the only two variables in the equation).
- Plot the point $(x, 0)$. This is called the $x$-intercept.
- Draw the line between the two intercepts.

*Your Turn!!*

Use the intercept method when graphing!

Graph $-x - 8y = -72$

Graph $2x + 3y = -15$

Graph $7x - y = 4$

Graph $6x - 10y = -50$
Graph $6x - 2y = 12$

Graph $-10x + y = -8$

Graph $-5x + 5y = -40$

Graph $9x + 3y = 15$

Graph $-5x + 6y = -48$

Graph $-10x + 5y = 25$
<table>
<thead>
<tr>
<th>Equation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-x + 4y = 0)</td>
<td><img src="image1" alt="Graph -x + 4y = 0" /></td>
</tr>
<tr>
<td>(x + 3y = 24)</td>
<td><img src="image2" alt="Graph x + 3y = 24" /></td>
</tr>
<tr>
<td>(6x - 6y = 30)</td>
<td><img src="image3" alt="Graph 6x - 6y = 30" /></td>
</tr>
<tr>
<td>(-3x + 4y = -28)</td>
<td><img src="image4" alt="Graph -3x + 4y = -28" /></td>
</tr>
<tr>
<td>(-2x - 4y = 20)</td>
<td><img src="image5" alt="Graph -2x - 4y = 20" /></td>
</tr>
<tr>
<td>(10x - 7y = -42)</td>
<td><img src="image6" alt="Graph 10x - 7y = -42" /></td>
</tr>
</tbody>
</table>
Graph 6: $6x - y = 1$

Graph 9: $9x - 2y = 6$

Graph 10: $10x - 6y = -18$

Graph 11: $x + 4y = 36$
Section 3.4: Graphing a Linear Inequality

**Big Skill:** You should be able to graph inequalities by drawing either a solid or dotted line, and then shading in one side of the line or the other.

**To graph a linear inequality:**
1. Graph the boundary line by solving for the dependent variable.
   a. If the inequality uses ≥ or ≤, then draw a solid line to show that the line itself satisfies the inequality.
   b. If the inequality uses just < or >, then draw a dashed line to show that the line does not satisfy the inequality.
2. Shade the appropriate side.
   a. If the inequality is < or ≤, shade the graph below the line.
   b. If the inequality is > or ≥, shade the graph above the line.

**Your Turn!!**
Graph the following inequalities.

Graph \(-4x - 6y \leq -18\)  

Graph \(6x - 5y \geq -5\)
<table>
<thead>
<tr>
<th>Graph</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4x + 6y &lt; -54</td>
<td>10x + 8y &lt; 40</td>
</tr>
<tr>
<td>-4x + 7y &lt; 21</td>
<td>-7x + 9y ≤ 18</td>
</tr>
<tr>
<td>4x + 9y ≤ -72</td>
<td>-8x – 8y ≤ -48</td>
</tr>
</tbody>
</table>
Graph $5x + 3y \leq 27$

Graph $-10x - 4y > 32$

Graph $-5x + 9y \geq 63$

Graph $10x + 5y < 10$

Graph $4x + 3y > 9$

Graph $7x + 4y \leq 28$
Section 3.5: Solving a System of Two Linear Equations by Graphing

**Big Skill:** You should be able to solve a system of two linear equations in two unknowns by graphing the lines and reading the intersection point off the graph.

**Example of the type of problem the skills in this chapter help us solve:**
Suppose that one country club has a $200 membership fee, and the golf costs $36 per round. A second country club has no membership fee and the golf costs $40 per round. How many games would you have to play at each club so that the cost was the same?
Let $n$ = the number of games you play
Let $c$ = the cost of playing that number of games
For club #1: $c = 36n + 200$
For club #2: $c = 40n$

➔ Graph the lines $y = 36x + 200$ and $y = 40x$ and see where they cross…

From the graph, it looks like if we play 50 games at either club, the cost will be $2,000.00.
A **system of linear equations** is a grouping of two or more linear equations, each of which contains the same variables.

**Examples:**

\[
\begin{align*}
y &= 36x + 200 \\
y &= 40x
\end{align*}
\]

\[
\begin{align*}
2x + 3y &= 4 \\
3x - y &= -5
\end{align*}
\]

In previous chapters, when we had just a single linear equation (in one variable) our goal was to find the **single** number that made the equation be true. That number was called the solution of the equation.

**Example:**

The equation \(2x - 5 = 15\) has a solution of \(x = 5\), because \(2(5) - 5 = 10 - 5 = 15\)

Now for this chapter, our goal is to find the **solution to a system of linear equations**, which consists of values for both of the variables \(x\) and \(y\) that make both equations in the system be true.

**Example:**

The system of equations

\[
\begin{align*}
y &= 36x + 200 \\
y &= 40x
\end{align*}
\]

has solution \((x, y) = (50, 2000)\) because:

\[
\begin{align*}
2000 &= 36(50) + 200 \\
2000 &= 40(50) \\
2000 &= 1800 + 200 \\
2000 &= 2000
\end{align*}
\]

**Solving a System of Two Linear Equations by Graphing**

- Graph both the lines.
- Read the coordinates of the intersection point off the graph.
- Check to see if those coordinates are the solution.

We see from these examples that there are three different cases for the solution to a system of two equations in two variables. We describe these cases using the words:

- **Consistent**, which means that there is at least one solution (no solutions \(\rightarrow\) **inconsistent**)
- **Dependent**, which means that the graphs of the lines are the same (different lines \(\rightarrow\) **independent**)

**Three Possible Cases for Solutions of a System of Two Linear Equations in Two Variables:**

**INTERSECT**: The lines intersect at one point, and thus the system has exactly one solution. This type of system is called **consistent** and the equations are called **independent**.

**PARALLEL**: The lines never intersect (i.e., they are parallel to one another), and thus the system has no solutions. This type of system is called **inconsistent** and the equations are called **independent**.

**COINCIDENT**: The lines lie on top of each other, and thus the system has infinitely many solutions. This type of system is called **consistent** and the equations are called **dependent**.

We can identify which of these three cases a system of equations will fall into (without graphing) by putting both equations into slope-intercept form, and comparing the slopes and \(y\)-intercepts.
Your Turn!!
Solve the following systems of equations by graphing. Verify your solutions by plugging them back into the original system.

1) \[
\begin{align*}
-8x + 4y &= 24 \\
4x - 2y &= -12
\end{align*}
\]

2) \[
\begin{align*}
2x - 7y &= 82 \\
-x - y &= 4
\end{align*}
\]

3) \[
\begin{align*}
-2x - 2y &= -14 \\
-4x + 4y &= -12
\end{align*}
\]
4) \[
3x - 4y = -12 \\
-3x - 2y = 30
\]
5) \[
\begin{align*}
6x + 5y &= 58 \\
-7x + 8y &= -40
\end{align*}
\]

6) \[
\begin{align*}
6x + 8y &= -40 \\
-2x + 4y &= -40
\end{align*}
\]
7) \[ \begin{align*}
-7x - 4y &= -91 \\
-7x + 3y &= -42
\end{align*} \]

8) \[ \begin{align*}
4x - y &= 27 \\
-7x + 6y &= -26
\end{align*} \]
9) \[
\begin{cases}
-2x - 3y = -11 \\
4x - 4y = -48
\end{cases}
\]

10) \[
\begin{cases}
6x - 7y = 73 \\
-4x - 8y = 40
\end{cases}
\]
Section 3.6: Solving a System of Two Linear Equations by Algebraic Methods

**Big Skill:** You should be able to solve a system of two linear equations in two unknowns by solving one equation for one of the variables, then replacing that variable in the other equation with its expression.

In the previous section, we saw that the lines \(\begin{cases} y = 36x + 200 \\ y = 40x \end{cases}\) cross at the point \((50, 2000)\), which means that if you play 50 games at either club, the cost will be $2,000. Here is how to solve this system using a purely algebraic technique called “substitution”.

Since the first equation already tells us that \(y = 36x + 200\), we can replace the \(y\) in the second equation with \(36x + 200\):

\[
\begin{align*}
y &= 40x \\
36x + 200 &= 40x \\
36x + 200 - 36x &= 40x - 36x \\
200 &= 4x \\
\frac{200}{4} &= \frac{4x}{4} \\
50 &= x 
\end{align*}
\]

Now that we know \(x = 50\), we can replace the \(x\) in \(y = 36x + 200\) to get:

\[
\begin{align*}
y &= 36x + 200 \\
&= 36(50) + 200 \\
&= 1800 + 200 \\
&= 2000 \\
y &= 2000 
\end{align*}
\]

So, we just derived that the solution to this system is \((50, 2000)\).

**Solving a System of Two Linear Equations Using the Substitution Method**

- Solve one of the equations for one of the variables; pick the easiest variable to solve for.
- Replace that variable in the other equation with the expression you just derived.
- Solve your new equation; it should have only one variable in it.
- Substitute that answer into the first equation and solve it to find the value of the original variable.
- Check to see if those coordinates are the solution.
Your Turn!!

Solve the following systems of equations using algebra. Verify your solutions by plugging them back into the original system AND by graphing the lines.

11) \[ \begin{align*}
-2x - y &= -16 \\
4x - y &= 14
\end{align*} \]

12) \[ \begin{align*}
-5x - 2y &= 38 \\
-7x - 2y &= 46
\end{align*} \]

13) \[ \begin{align*}
4x + 8y &= 36 \\
-2x + 7y &= 37
\end{align*} \]
14) \[ \begin{align*}
8x + 7y &= -45 \\
2x - 8y &= 18
\end{align*} \]
15) \[ \begin{cases} x - y = 5 \\ 3x - 7y = 7 \end{cases} \]

16) \[ \begin{cases} -4x + 8y = 8 \\ 5x - 6y = -6 \end{cases} \]
17) \[
\begin{align*}
-3x + 5y &= 57 \\
6x + 3y &= 3
\end{align*}
\]

18) \[
\begin{align*}
3x - 3y &= 0 \\
-5x - 4y &= -27
\end{align*}
\]
19) \[ \begin{align*}
5x - 5y &= -35 \\
-4x - 6y &= -52
\end{align*} \]

20) \[ \begin{align*}
-3x + 3y &= 15 \\
-x + 3y &= 17
\end{align*} \]
Chapter 3 Sample Exam

1. ( __ / 15 points ) Solve: \(2(2k - 3) - 4(6k - 7) = -3(3k - 9)\)

2. ( __ / 12 points ) Solve for \(L\): \(A = 2L(W + H) + 2WH\)

3. ( __ / 15 points ) Solve the inequality, then graph the solution: \(-2(4x - 3) + 8 \leq -\frac{1}{2}(4x + 6) - 1\)
4. ( __ / 20 points ) A school district has three pay scales. The lowest pay scale is $35,000 per year. The middle pay scale is 10% more per year, and there are twice as many teachers at that middle pay scale than at the lowest pay scale. The highest pay scale is 15% more than the middle pay scale, and there are ten less teachers at the highest pay scale than at the middle pay scale. The district pays $7,579,250.00 in total salary every year. Compute how many teachers are in the district.

5. ( __ / 10 points ) The price paid for a TV is $949.45 with the 5.5% sales tax added on. Compute the sticker price (i.e., before tax price) of the TV.

6. ( __ / 15 points ) Compute how much you can borrow if you can afford a monthly payment of $250, and you can get a five-year loan at a 4.5% annual interest rate.
7. ( ___ / 15 points ) The distance a car’s shock absorber compresses is directly proportional to how much force is applied to it. If the shock absorber compresses a distance of 15.0 mm when a force of 1,215 pounds is applied, compute the compression when a force of 1,982 pounds is applied. Use the correct number of significant figures in your answer.

8. ( ___ / 15 points ) The height of a ripple on a pond caused by throwing in a pebble is inversely proportional to the distance of the ripple from the pebble. If the ripple height is 0.859 inches at a distance of 1.00 meters from the pebble, find the height of the ripple when it is 3.79 meters from the pebble. Use the correct number of significant figures in your answer.

9. ( ___ / 10 points ) Convert 1.75% to a proper fraction reduced to lowest form.
10. ( ___ / 15 points ) Suppose that Cell Phone Company #1 charges a flat fee of $10 per month plus 10 cents per minute of calls, and that Cell Phone Company #2 charges a flat fee of $15 per month plus 8 cents per minute of calls. Use a linear equality to state how many minutes you can talk for Cell Phone Company #2 to be cheaper for the month.

11. Extra Credit (6 points): If the price of a restaurant meal is $66.37 after the 5.5% sales tax and 15% tip is added on, find the price of the meal before those add-ons.
Chapter 4: Measurement

Section 4.1: Linear Measurements

Measurement conversion is a necessary skill since the same set of units is often not used throughout a calculation. The basis of measurement conversion is the unit fraction.

A unit fraction is a fraction that has a value of 1. The catch is that 1 has infinitely many “aliases”. For example,

\[
\frac{12 \text{ in}}{1 \text{ ft}} = \frac{1 \text{ ft}}{12 \text{ in}} = \frac{4 \text{ qt}}{1 \text{ gal}} = \frac{1 \text{ gal}}{4 \text{ qt}} = \frac{1 \text{ min}}{60 \text{ sec}} = \frac{60 \text{ sec}}{1 \text{ min}} = \frac{1000 \text{ m}}{1 \text{ km}} = \frac{1 \text{ km}}{1000 \text{ m}}
\]

All of these represent unit fractions, since the numerator is the same amount as the denominator.

When multiplying a quantity by a unit fraction (which has a value of one), the value of the quantity remains unchanged. That is for any quantity \( Q \):

\[
Q = Q \cdot 1
\]

The trick is to use the “proper” aliases to cancel the units you don’t want and get the units you do want. For example, to convert 18 in to ft we use the unit fraction containing the ratio of the number of inches in a foot as follows:

\[
18 \text{ in} = \frac{18 \text{ in}}{1} \times \frac{1 \text{ ft}}{12 \text{ in}} = \frac{18}{12} \text{ ft} = \frac{3}{2} \text{ ft} = 1 \frac{1}{2} \text{ ft or } 1.5 \text{ ft}
\]

As you can see, the number of inches is placed in the denominator of the unit fraction so that it appears in both the numerator and denominator and as a result cancels out.

Section 4.1.1: Conversions within the English System

<table>
<thead>
<tr>
<th>Linear Measure</th>
<th>Area Measure</th>
<th>Volume Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ft = 12 in</td>
<td>1 acre = 43,560 sq ft</td>
<td>1 pt = 16 oz</td>
</tr>
<tr>
<td>1 yd = 3 ft</td>
<td>1 acre = 160 sq rods</td>
<td>1 qt = 2 pt = 32 oz</td>
</tr>
<tr>
<td>1 mi = 5,280 ft</td>
<td>1 sq mile = 640 acres</td>
<td>1 gal = 4 qt = 128 oz</td>
</tr>
<tr>
<td>1 mi = 1,760 yd</td>
<td></td>
<td>1 gal = 231 in³</td>
</tr>
<tr>
<td>1 rod = 16.5 ft</td>
<td></td>
<td>1 ft³ = 7,480 519 gal</td>
</tr>
<tr>
<td>1 furlong = 220 yd</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Weight</th>
<th>Time</th>
<th>Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 lb = 16 oz</td>
<td>1 min = 60 s</td>
<td>1 mi / hr = 1.466 467 ft / sec</td>
</tr>
<tr>
<td>1 ton = 2,000 lb</td>
<td>1 hr = 60 min</td>
<td></td>
</tr>
<tr>
<td>1 stone = 14 lb</td>
<td>1 hr = 3600 s</td>
<td></td>
</tr>
<tr>
<td>1 slug = 32.174 049 lb</td>
<td>1 day = 24 hours</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 day = 1,440 min</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 day = 86,400 sec</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 year = 365.25 day</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 year = 31,557,600 sec</td>
<td></td>
</tr>
</tbody>
</table>
As mentioned at the beginning of the chapter we will look at conversions within the English System and the Metric System. In this section, as in the example above, we will continue to look at conversions within the English System.

**Example:** *Convert 5 gallons to quarts.*

**Solution:** First, we must identify the necessary conversion factor between gallons and quarts. Since there are 4 quarts in 1 gallon, we set up a unit fraction using these values. We place the 1 gallon in the denominator of our unit fraction as follows because we are trying to “get rid of” the gallon units.

\[
5 \text{ gallons} = \frac{5 \text{ gallons}}{1} \times \frac{4 \text{ quarts}}{1 \text{ gallon}} = 20 \text{ quarts}
\]

Thus 5 gallons is equivalent to 20 quarts.

As an aide in setting up conversion calculations, a set of equivalent measurements is presented in the appendix at the end of this chapter.

There are times when we wish to convert measurements that are in decimal form to fractional form. The same technique using unit fractions works.

**Example:** *Round 0.567 in to the nearest 32\textsuperscript{nd} of an inch.*

**Solution:** Since we are trying to convert to 32\textsuperscript{nd} of an inch, the key is to use the unit fraction \(\frac{32}{32}\).

\[
0.567 \text{ in} = \frac{0.567 \text{ in}}{1} \times \frac{32}{32} = \frac{0.567 \times 32}{1 \times 32} = \frac{18.144}{32} \text{ in} \approx \frac{18}{32} \text{ in}
\]

0.567 inches is eighteen thirty-seconds to the nearest thirty-seconds of an inch.

**Example:** *Round 0.434 in to the nearest 64\textsuperscript{th} of an inch.*

**Solution:** Since we are trying to convert to 64\textsuperscript{th} of an inch, the key is to use the unit fraction \(\frac{64}{64}\).

\[
0.434 \text{ in} = \frac{0.434 \text{ in}}{1} \times \frac{64}{64} = \frac{0.434 \times 64}{64} \text{ in} = \frac{27.776}{64} \text{ in} \approx \frac{28}{64} \text{ in} = \frac{7}{16} \text{ in}
\]

So 0.434 in is seven sixteenth’s of an inch to the nearest 64\textsuperscript{th} of an inch.

For some problems we do not have a conversion factor that directly gives us the number of one type of unit in terms of the other. In these cases we may need to multiply by more than one unit fraction to get us from one unit to the other.

**Example:** *Convert 190 ounces to gallons.*

**Solution:**

\[
190 \text{ oz} = \frac{190}{1} \times \frac{1 \text{ pt}}{16 \text{ oz}} \times \frac{1 \text{ gal}}{2 \text{ pt}} = \frac{190 \times 1 \times 1}{1 \times 16 \times 2} \text{ gal}
\]

Madison College’s College Mathematics Textbook  
Page 124 of 256
= \frac{190}{128} \text{ gal} = \frac{95}{64} \text{ gal} = \frac{31}{64} \text{ gal} = 1.484375 \text{ gal}

190 ounces is \frac{31}{64} \text{ gallons} or 1.484375 \text{ gallons} which is approximately 1.5 gallons.

In other more complicated conversions we may need to use more than one unit fraction because we have a measurement involving some type of rate where we need to convert both units involved in the rate.

Example: Convert 100 feet per second to miles per hour correct to one decimal place.

Solution: In this problem, we not only multiply by the unit fraction to convert the feet units to miles, but we must also multiply by the unit fractions that convert seconds to hours. As we do this we must make sure to set up the unit fractions so that the units we wish to “get rid of” are diagonal from each other so that they cancel.

\[
100 \text{ ft/sec} = \frac{100 \text{ ft}}{1 \text{ sec}} \times \frac{1 \text{ mile}}{5280 \text{ ft}} \times \frac{60 \text{ sec}}{1 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hr}} = \frac{100 \times 1 \times 60 \times 60}{1 \times 5280 \times 1 \times 1} \text{ mile/hr}
\]

\[
= \frac{360,000 \text{ mi}}{5280 \text{ hr}} = \frac{2250 \text{ mi}}{33 \text{ hr}} = 68.6 \frac{\text{mi}}{\text{hr}} \approx 68.2 \frac{\text{mi}}{\text{hr}}
\]

Section 4.1.2: The Metric System

Conversions within the metric system simply require the shifting of the decimal point.

One of the consequences of the French Revolution of 1789 was the development of the metric system of measurement. This system was designed to replace the earlier French system, which like its English counterpart had its origins in medieval society and royal institutions. Three features make the metric system very attractive. First, it is built on powers of 10, just like our decimal number system. Every unit is a multiple of 10 of some other unit. Thus, “strange” English multipliers like 3, 12, and 16 are banished! Second, a deliberate effort was made to coordinate different measures. For example, the fundamental unit of volume, the liter symbolized by L, is simply related to the fundamental unit of length, the meter symbolized by m, through the equation 1 m³ = 1000 L. Contrast this with the English system where 1 gal = 231 in³ = 0.134 ft³. The third advantage of the metric system is that it is “universal”. It can be used with any kind of measurement in the same way.

It is interesting to note that the metric system was so well accepted and in place when electrical measurements began some 150 years ago, only metric units were developed. The customary electric units we are all know, the volt (V), amp (A), and ohm (Ω) are all metric.

The metric system uses a two-part representation for all measurements. Each unit of measurement contains the base unit which is determined by what we are trying to measure. The important metric base units are as follows:

<table>
<thead>
<tr>
<th>Base Unit</th>
<th>Symbol</th>
<th>What it Measures</th>
</tr>
</thead>
<tbody>
<tr>
<td>meter</td>
<td>m</td>
<td>length</td>
</tr>
<tr>
<td>seconds</td>
<td>s</td>
<td>time</td>
</tr>
<tr>
<td>gram</td>
<td>g</td>
<td>mass</td>
</tr>
</tbody>
</table>
### Other units smaller or larger than the base unit are created by multiplying or dividing the base unit by powers of 10. The names of each of these smaller/larger units are identified by attaching a prefix to the name of the base unit. The first character or prefix indicates the power of 10 of the number of the base units being used.

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Symbol</th>
<th>Power of Ten</th>
<th>Number per Base Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>nano</td>
<td>n</td>
<td>$10^{-9} = 0.000000001$</td>
<td>$10^9 = 1,000,000,000$</td>
</tr>
<tr>
<td>micro</td>
<td>$\mu$</td>
<td>$10^{-6} = 0.000001$</td>
<td>$10^6 = 1,000,000$</td>
</tr>
<tr>
<td>milli</td>
<td>m</td>
<td>$10^{-3} = 0.001$</td>
<td>$10^3 = 1000$</td>
</tr>
<tr>
<td>centi</td>
<td>c</td>
<td>$10^{-2} = 0.01$</td>
<td>$10^2 = 100$</td>
</tr>
<tr>
<td>deci</td>
<td>d</td>
<td>$10^{-1} = 0.1$</td>
<td>$10^1 = 10$</td>
</tr>
<tr>
<td>deca</td>
<td>da</td>
<td>$10^1 = 10$</td>
<td>$10^{-1} = \frac{1}{10} = 0.1$</td>
</tr>
<tr>
<td>hecto</td>
<td>h</td>
<td>$10^2 = 100$</td>
<td>$10^{-2} = \frac{1}{100} = 0.01$</td>
</tr>
<tr>
<td>kilo</td>
<td>k</td>
<td>$10^3 = 1000$</td>
<td>$10^{-3} = \frac{1}{1000} = 0.001$</td>
</tr>
<tr>
<td>mega</td>
<td>M</td>
<td>$10^6 = 1,000,000$</td>
<td>$10^{-6} = \frac{1}{1,000,000}$</td>
</tr>
<tr>
<td>giga</td>
<td>G</td>
<td>$10^9 = 1,000,000,000$</td>
<td>$10^{-9} = \frac{1}{1,000,000,000} = 0.000000001$</td>
</tr>
</tbody>
</table>

### Examples of Metric Units:

**Centimeter (cm):** Centimeters are a smaller unit than the meter. Centimeters are $10^{-2}$ or $\frac{1}{100}$ of a meter. This means there are $10^2$ or 100 centimeters in one meter.

**Kilohertz (kHz):** Kilohertz are a larger unit than the hertz. A kilohertz is $10^3$ or 1000 hertz. This means that there is $10^{-3}$ or $\frac{1}{1000}$ of a kilohertz in one hertz.

**Milligrams (mg):** Milligrams are a smaller unit than the gram. Milligrams are $10^{-3}$ of a gram. This means there are $10^3$ or 1000 milligrams in one gram.
Conversions within the Metric System:
Conversions within the metric system simply require the shifting of the decimal point. As with conversions within the English System, we can use the idea of unit fractions to help us convert between units in the Metric System. We create unit fractions by using what the prefix means in terms of the base unit.

Example: Convert 0.264 kilograms to grams.

Solution: \[
0.264 \text{ kg} \times \frac{10^3 \text{ g}}{1 \text{ kg}} = 264 \text{ g}
\]

Notice that in the example above, we shift the decimal point three places to the right since we are multiplying by \(10^3\).

Example: Convert 175 millimeters to centimeters.

Solution: \[
175 \text{ mm} \times \frac{1 \text{ cm}}{10 \text{ mm}} = 17.5 \text{ cm}
\]

Notice in this example, we end up shifting 1 decimal place to the left since we need to divide by 10.

Example: Convert 0.00059 kA to mA.

Solution: \[
0.00059 \text{ kA} \times \frac{10^3 \text{ A}}{1 \text{ kA}} \times \frac{10^3 \text{ mA}}{1 \text{ A}} = 590 \text{ mA}
\]

Notice in this example, we end up shifting the decimal point 6 places to the right since we are multiplying by \(10^6\).

Section 4.1.3: Performing Operations with Measurements

In many situations we may need to perform operations with measurements. We may need to add measurements to find the perimeter around an object; we may need to subtract to find out how much we have left; we may want to multiply measurements to find area or the amount of work required. In this section we take a look at how to perform these operations.

Section 4.1.3a: Addition and Subtraction of Measurements:

It is not sensible to add or subtract measurements of different kinds of things. For example, 15 lb + 7 ft is a meaningless operation. This is just the old adage that it’s impossible to add apples and oranges! To add or subtract measurements requires the same kind of quantities, as in 9 feet + 8 feet = 17 feet. Note, we just add the numbers and carry the factor of the unit. This is just the distributive property discussed in the Algebra Chapter. In some problems there may be two different units involved in each measurement. In these problems the key is to add the like type of units.

Example: Perform the indicated operation: 9 lb 8 oz + 4 lb 9 oz
9 lb 8 oz
Solution:  
+ 4 lb 9 oz
13 lb 17 oz

However since 17 oz is more than 1 pound’s worth of ounces, we do not leave the answer in this form. We take 16 of the ounces and convert them to one pound. This leaves us with one ounce. We then add one pound to our current total number of pounds and we write the remaining number of ounces.

13 lb 17 oz = 13 lb + 16 oz + 1 oz = 13 lb + 1 lb 10 oz = 14 lb 1 oz

Our final answer is: 14 lb 1 oz.

What do we do, however, if the units of the measurements we are asked to add or subtract are not the same? We must do unit conversions so that both measurements have the same units.

Example: Perform the indicated operation: 8 ft + 36 in.

Solution: The quantities to be added are both lengths so the operation makes sense, but we can’t actually perform the addition until we get the units to agree. We must either get both units to be inches or both to be feet.

Performing the indicated operation using feet:
We must first convert the 36 inches to feet as follows:

\[
36 \text{ in} = \frac{36 \text{ in}}{1} \times \frac{1 \text{ ft}}{12 \text{ in}} = \frac{36}{12} \text{ ft} = 3 \text{ ft}
\]

Now we perform the operation using 3 feet in place of the 36 inches.

8 ft + 3 ft = 11 ft
Thus 8 ft + 36 in is 11 feet.

Performing the indicated operation using inches:
First we must convert 8 feet to inches as follows:

\[
8 \text{ ft} = \frac{8 \text{ ft}}{1} \times \frac{12 \text{ in}}{1 \text{ ft}} = 96 \text{ in}
\]

Now we perform the operation using 96 inches in place of 8 feet.

8 ft + 36 in = 96 in + 36 in = 132 in
Thus an alternate answer to this problem is that 8 ft + 36 in is equal to 132 inches.

Example: Perform the indicated operation: 4 ft – 14 in.

Solution: First we convert the 4 feet to inches.

\[
\frac{4 \text{ ft}}{1} \times \frac{12 \text{ in}}{1 \text{ ft}} = 48 \text{ in}
\]

Next we replace the 4 ft by 48 in and do the subtraction.

4 ft – 14 in = 48 in – 14 in = 34 in
We can write the answer in one of two ways. We can write the answer as 34 inches; however, it may be preferable to have the answer in terms of feet and inches. To convert this into feet and inches we divide 34 by 12 and we get 2 full feet (24 inches) and write the remaining number of inches.

\[
\frac{34 \text{ in}}{1} \times \frac{1 \text{ ft}}{12 \text{ in}} = 2 \text{ ft} 10 \text{ in}
\]

Thus an alternate answer to this problem is 2 feet and 10 inches.

**Example:**

*Perform the indicated operation:* \((5 \text{ tons } 250 \text{ lb}) - (2 \text{ ton } 1225 \text{ lb})\)

**Solution:**

If we set up the problem vertically as follows, we notice that we do not have enough pounds in the first measurement to do the subtraction.

\[\begin{array}{c}
5 \text{ tons } 250 \text{ lb} \\
- \quad 2 \text{ ton } 1225 \text{ lb}
\end{array}\]

As a result, we must either borrow or convert all of the units to pounds and then do the subtraction. When borrowing, we borrow 1 from the tons column and convert it over to its equivalent value of 2000 lb. We add the 2000 lb to the number of pound we already have and do the subtraction.

\[\begin{array}{c}
\text{4 tons} \\
250 + 2000 \text{ lb}
\end{array}\]

\[\begin{array}{c}
- \quad 2 \text{ tons } 1225 \text{ lb} \\
2 \text{ tons } 1025 \text{ lb}
\end{array}\]

Thus the answer is 2 tons 1025 lb.

**Section 4.1.3b Multiplying and Dividing Measurements:**

While adding or subtracting different kinds of measurements is impossible, multiplying or dividing measurements is always possible. As mentioned above there are two parts to every measurement. If we are asked to multiply/ divide two measurements we not only multiply/ divide the number parts of the measurements, we also multiply/ divide the unit parts of each measurement.

**Example:**

*Perform the indicated operation:* \(5 \text{ lb } \times 4 \text{ ft}\)

**Solution:**

\[5 \text{ lb } \times 4 \text{ ft} = (5 \times 4)(\text{lb } \times \text{ ft}) = 20 \text{ ft } \cdot \text{ lb}\] where a \(\text{ ft } \cdot \text{ lb}\) is 1 ft \(\times\) 1 lb is a “foot pound” which is a measure of either energy or torque.

**Example:**

*Perform the indicated operation:* \(100 \text{ miles } \div 4 \text{ gallons}\)

**Solution:**

\[100 \text{ miles } \div 4 \text{ gallons} = \frac{100}{4} \frac{\text{ miles}}{\text{ gallons}} = 25 \text{ miles per gallon}\]

**Example:**

*Perform the indicated operation:* \($3.95 \text{ per gallon } \div 32 \text{ miles per gallon}\)

**Solution:**

\[\frac{3.95}{1 \text{ gallon}} \div 32 \text{ miles} = \frac{3.95}{1 \text{ gallon}} \times \frac{1 \text{ gallon}}{32 \text{ miles}} = 0.1234375 \text{ miles } \approx 0.12 / \text{ mile}\]

**Example:**

*Perform the indicated operation:* \((12 \text{ in})^2\)
Solution: \[(12\text{ in})^2 = (12\text{ in})(12\text{ in}) = (12\times12)(\text{in\times in}) = 144 \text{ in}^2\]

Your Turn!!

1. Convert 60 inches to feet.  
   1) ________________________________

2. Convert 12 yards to feet.  
   2) ________________________________

3. Convert 42,240 feet to miles.  
   3) ________________________________

4. Convert 40 feet to yards and feet.  
   4) ________________________________

5. Convert 5 feet 2 inches to inches.  
   5) ________________________________

6. Convert 12,000 pounds to tons.  
   6) ________________________________

7. Round to the nearest 32'nd of an inch: 0.165 in.  
   7) ________________________________

8. Round to the nearest 64'th of an inch: 0.645 in  
   8) ________________________________

9. What size bolt, to the nearest 64th of an inch, will fit a hole 0.59 in diameter?  
   9) ________________________________

10. How many micrograms are in 1 gram?  
    10) ________________________________

11. How many joules are in a gigajoule?  
    11) ________________________________

12. How many decimeters are in 1 meter?  
    12) ________________________________

13. How many nanoseconds are in 1 second?  
    13) ________________________________

14. Which is a larger unit: volt or kilovolt?  
    14) ________________________________
15. Which is a larger unit: milliliter or centiliter? 15) ________________________________

16. Convert 45.5 m to cm. 16) ________________________________

17. Convert 40 mm to cm. 17) ________________________________

18. Convert 355 m to km. 18) ________________________________

19. Convert 1400 mm to meters. 19) ________________________________

20. Convert 1585 cm to meters. 20) ________________________________

21. Convert 8.3 cm to mm. 21) ________________________________

22. Convert 525 grams to kilograms. 22) ________________________________

23. Convert 4.5 grams to milligrams. 23) ________________________________

24. Convert 48.2 mg to grams. 24) ________________________________

25. Convert 6.3 grams to kilograms. 25) ________________________________

26. Convert 4.01 kg to grams. 26) ________________________________

27. Convert 4500 ml to Liters. 27) ________________________________

28. Convert 410 volts to kilovolts. 28) ________________________________

29. Convert 0.16 kilowatts to watts. 29) ________________________________

30. Convert 3.6 liters to milliliters 30) ________________________________
31. Convert 260 mA to A. 

32. Perform the indicated operation: 12 ft 7 in + 8 ft 8 in

33. Perform the indicated operation: 4 yd + 7 ft

34. Perform the indicated operation: 49 in − 2 ft

35. Perform the indicated operation: (29 lb 3 oz) − (8 lb 15 oz)

36. Perform the indicated operation: 28 oz + 3 gal

37. Perform the indicated operation: 5 lb. 6 0z. − 2 lb. 9 oz.

38. Perform the indicated operation: 6 ton 1540 lb + 2 tons 850 lb

39. Perform the indicated operation: 4.2 cm × 3.5 cm²

40. Perform the indicated operation: (12 in)³

41. Perform the indicated operation: (3 ft)²

42. Perform the indicated operation: 465 miles ÷ 23 mpg
43. Perform the indicated operation: 590 miles ÷ 56 mi/hr

44. A 3.4 m rope is attached to a 5.8 m rope. However when the ropes are tied, 8 cm of length is lost to form the knot. What is the length of the tied ropes?

45. The ice on Tiedmann’s pond is 5.33 cm thick. For safe skating the town of Middleton requires 80 mm of ice thickness. How much thicker must the ice be in order for the pond to be available for skating?

46. One bag of Pepperidge Farm Bordeaux cookies weighs \( \frac{6 \frac{3}{4}}{4} \) ounces. How many pounds will a dozen bags weigh?

47. A can of 7-Up weights 336 grams. Determine the weight in kilograms of a case of 24 cans.

48. A floor tile is 22.86 cm wide. How many tiles will be in a row if the distance across the room is 3.429 m.

49. Stanley paid $14.00 to fill his car with 44.3 liters of gasoline. What is the price per liter of gasoline to the nearest cent?

50. If the price per barrel of crude oil is $127 and each barrel contains 31 gallons, how much per gallon is crude oil?

---

Section 4.2: Measuring Area

- The measurements in the previous section all involved single “fundamental” units for length, weight, and time (and electrical charge…).
- All other measurement units are “compound units” made up of factors of the four fundamental units.
- The units of measurement for area are some of the most basic compound units there are.
- Area is used to measure the amount of space occupied by a two-dimensional shape.
- The idea of area allows us to describe with one number the “amount contained” by a piece of cloth, a piece of sheet metal, a computer screen, a wound, or a piece of land.
• Remember that measurement involves deciding on a unit of measurement, and then counting up how many of those units are in the object we want to measure.
• To measure area, we use square units.
• A square unit is a square whose sides are each one unit in length.
• For example, if we choose to measure area by with a square that is one foot on each side as our unit of measurement, then the area of a 2 ft by 6 ft rectangle is 12 square feet, because exactly twelve of the one-foot by one-foot unit squares fit in the 2 ft by 6 ft rectangle.

![Diagram of One Square Foot and Twelve Square Feet]

- We called this unit measurement a square foot because it is a square that is one foot on each side.
- However, there is another, more algebraic, name for this unit. It is: 1 ft².
- That is because we must also be able to connect this unit to the geometry formula for the area of a square:
  \[ A = \text{(side length)} \cdot \text{(side length)} = (1 \text{ ft}) \cdot (1 \text{ ft}) = (1)(1)(\text{ft})(\text{ft}) = 1 \text{ ft}^2 \]
- In general, when converting area units, you have to square the conversion fraction because the units of measurement are a dimension squared.

Up to this point we have looked at measurements that involve length, weight, and time. In many situations we need to be able to describe the size of a piece of cloth, of a piece of sheet metal, of a computer screen, of a wound, or of a piece of land. In this section we take a look at how to measure the amount of space inside a two-dimensional figure. The amount of “two-dimensional space” inside of a planar figure is called its area. How do we measure “two-dimensional space?” We use square units. A square unit is a square whose sides are each one unit in length. For example, a 2 ft by 6 ft rectangle has an area of \( A = (\text{length})(\text{width}) = (2 \text{ ft})(6 \text{ ft}) = 12 \text{ ft}^2 \). What does this mean? 12 one foot by one foot squares will fit inside this region.

Care must be taken when converting units of area. Suppose we want to calculate how many square inches are in an area of 1.6 ft². In order to do this conversion, we need to know the number of square inches in a square foot.

How is this determined?
\[
1 \text{ ft}^2 = (1 \text{ ft})(1 \text{ ft}) = (1 \text{ ft})^2 = (12 \text{ in})^2 = 12^2 \text{ in}^2 = 144 \text{ in}^2
\]

Note: when we evaluate \((12 \text{ in})^2\) we square both the number (12) and the units (in). This is illustrated below.
Example: Convert 2.6 ft to in.

Solution: To perform the conversion we must multiply by the unit fraction \( \frac{(12 \text{ in})^2}{(1 \text{ ft})^2} \) or \( \frac{144 \text{ in}^2}{1 \text{ ft}^2} \).

\[
1.6 \text{ ft}^2 = 1.6 \frac{\text{ft}^2}{1} \times \frac{(12 \text{ in})^2}{(1 \text{ ft})^2} = \frac{1.6}{1} \times \frac{144 \text{ in}^2}{1 \text{ ft}^2} = 230.4 \text{ in}^2
\]

Thus there are 230.4 square inches in 1.6 square feet.

As mentioned above, we must be very careful when converting units of area. We must square the linear conversion factor to obtain the conversion factor for the square units used to measure area.

Example: Convert 35,000,000 ft to square miles.

Solution:

\[
35,000,000 \text{ ft}^2 = \frac{35,000,000 \text{ ft}^2}{1} \times \frac{(1 \text{ mi})^2}{(5280 \text{ ft})^2} = \frac{35,000,000 \text{ ft}^2}{1} \times \frac{1 \text{ mi}^2}{27,878,400 \text{ ft}^2}
\]

\[
= 1.25545225 \text{ mi}^2 \approx 1.3 \text{ mi}^2
\]

There are 1.25545225 square miles or approximately 1.3 square miles in 35,000,000 square feet.
So far we have just looked at examples involving units from the English System of measurement. The same process works when working with units of area in the Metric System.

Example: Convert 0.042 m² to cm².

Solution: 

\[0.042 \text{ m}^2 = \frac{0.042 \text{ m}^2}{1} \times \left(\frac{10^2 \text{ cm}}{1 \text{ m}}\right)^2 = 0.042 \text{ m}^2 \times \frac{10^4 \text{ cm}^2}{1 \text{ m}^2} = 420 \text{ cm}^2\]

Notice that the decimal point is shifted 4 places to the right since we are multiplying by \(10^4\).

Let’s turn to an application problem.

Example: A fire has spread to the entire floor of a building. If the floor of the building is 55 feet by 48 feet, how many square yards does the fire cover?

Solution: There are two methods for solving this problem. We can either convert each of the dimensions to yards before calculating the area, or we can calculate the area and then convert the square feet to square yards.

Solving by Calculating the Area First:

Area = (length)(width) = (55 ft)(48 ft) = 2640 ft²

Now we convert the square feet to square yards as follows:

\[2640 \text{ ft}^2 = \frac{2640 \text{ ft}^2}{1} \times \left(\frac{1 \text{ yd}^2}{3 \text{ ft}^2}\right)^2 = \frac{2640 \text{ ft}^2}{1} \times \frac{1 \text{ yd}^2}{9 \text{ ft}^2} = \frac{2640}{9} \text{ yd}^2 = 293.3 \text{ yd}^2 \approx 293.3 \text{ yd}^2\]

The fire is covering \(293\frac{1}{3}\) yd² or approximately 293.3 yd².

Solving by Converting First:

First we convert each dimension as follows:

\[55 \text{ ft} = \frac{55 \text{ ft}}{1} \times \frac{1 \text{ yd}}{3 \text{ ft}} = \frac{55}{3} \text{ yd}\]

\[48 \text{ ft} = \frac{48 \text{ ft}}{1} \times \frac{1 \text{ yd}}{3 \text{ ft}} = \frac{48}{3} \text{ yd} = 16 \text{ yd}\]

Next we calculate the area:

\[Area = (\text{length})(\text{width}) = \left(\frac{55}{3} \text{ yd}\right)(16 \text{ yd}) = \frac{55 \times 16}{3} \text{ yd}^2 = \frac{880}{3} \text{ yd}^2 = 293\frac{1}{3} \text{ yd}^2\]

As found when calculating the area first, the fire is covering \(293\frac{1}{3}\) yd² or approximately 293.3 yd².

Your Turn!!

1. Convert 0.82 ft² to in²

   1) ___________________________
2. Convert 329 in² to ft².

3. Convert 925 ft² to yd².

4. Convert 1.24 yd² to ft².

5. Convert 0.35 mi² to ft².

6. Convert 179,000,000 ft² to mi².

7. Convert 0.46 cm² to mm².

8. Convert 1239 cm² to m².

9. Convert 12,325 m² to km².

10. A piece of sheet metal is 36 inches by 27 inches. How many square feet of sheet metal are there?

11. A wound is circular in shape. If the radius of the wound is 0.6 cm and area of a circle can be found by 
\[\text{Area} = \pi \times \text{(radius)}^2\] (where \(\pi \approx 3.14\)), approximate the area of the wound to the nearest square millimeter.

10) __________________________

11) __________________________
Section 4.3: Measuring Volume

- Remember that measurement involves deciding on a unit of measurement, and then counting up how many of those units are in the object we want to measure.
- To measure area, we use cubic units.
- A cubic unit is a cube whose sides are each one unit in length.
- For example, if we choose to measure volume with a cube that is one foot on each side as our unit of measurement, then the volume of a cube that is one yard on a side is 27 cubic feet, because exactly 27 of the one-foot by one-foot by one-foot unit cubes fit in the 3 ft by 3 ft by 3 ft cube.

![Diagram of cubic foot and cubic yard]

- We called this unit measurement a cubic foot because it is a cube that is one foot on each side.
- However, there is another, more algebraic, name for this unit. It is: 1 ft³.
- That is because we must also be able to connect this unit to the geometry formula for the volume of a cube:
  - \[ V = (\text{side length})(\text{side length})(\text{side length}) \]
  - \[ = (1 \text{ ft})(1 \text{ ft})(1 \text{ ft}) \]
  - \[ = (1)(1)(1)(\text{ft})(\text{ft})(\text{ft}) \]
  - \[ = 1 \text{ ft}^3 \]
- In general, when converting area units, you have to cube the conversion fraction because the units of measurement are a dimension cubed.

Just as we often want to measure the space inside a two-dimensional object, we have reason to measure the space inside a three-dimensional object. We may want to measure the space an engine takes up, the space inside a box, the amount of water we can fit inside a tank, or how much space we have for blood in a vial. **Volume** is the amount of “three-dimensional space” inside of a solid. How do we measure space inside a three-dimensional object? We use cubic units. A cubic unit is a cube whose length, width and height each
have a length of 1 unit. For example, a 2 ft by 3 ft by 2 ft box has a volume, \( V = (\text{length})(\text{width})(\text{height}) = (2 \text{ ft})(3 \text{ ft})(2 \text{ ft}) = 12 \text{ ft}^3 \). Here the unit \( \text{ft}^3 \) is one cubic foot (cu ft), which literally means a one foot by one foot by one foot cube as shown below.

![Cubic Foot Diagram]

When we say that the volume of the box is 12 \( \text{ft}^3 \), we mean that we could fit exactly 12 one foot by one foot by one foot cubes inside this box.

Like area conversions, volume conversions require careful setup. Suppose we wish to convert cubic feet to cubic yards. We must first determine the conversion factor between these units. To do this we use the same kind of technique as we did when determining the conversion factor between square units.

\[
1 \text{ yd}^3 = (1 \text{ yd})(1 \text{ yd})(1 \text{ yd}) = (1 \text{ yd})^3 = (3 \text{ ft})^3 = 3^3 \text{ft}^3 = 27 \text{ ft}^3
\]

**Note:** when we evaluate \((3 \text{ ft})^3\) we cube both the number (3) and the units (ft). This is illustrated below.

![Cubic Yard Diagram]

**Example:** Convert 12 \( \text{ft}^3 \) to cubic yards.

**Solution:**

\[
12 \text{ ft}^3 = 
\frac{12 \text{ ft}^3}{1} \times \frac{(1 \text{ yd})^3}{(3 \text{ ft})^3} = 
\frac{12 \text{ ft}^3}{1} \times \frac{1 \text{ yd}^3}{27 \text{ ft}^3} = 
\frac{12}{27} \text{ yd}^3 \approx 0.44 \text{ yd}^3
\]

**Example:** Convert 2500 \( \text{in}^3 \) to \( \text{ft}^3 \).

**Solution:**

\[
2500 \text{ in}^3 = 
\frac{2500 \text{ in}^3}{1} \times \frac{(1 \text{ ft})^3}{(12 \text{ in})^3} = 
\frac{2500 \text{ in}^3}{1} \times \frac{1 \text{ ft}^3}{1728 \text{ in}^3} = 
\frac{2500}{1728} \text{ ft}^3 = 
\frac{1.93}{432} \text{ ft}^3 \approx 1.45 \text{ ft}^3
\]
Thus 2500 cubic inches is about 1.45 cubic feet.

As before, the same process works for metric units as well.

**Example:** Convert 2695 mm$^3$ to cm$^3$.

**Solution:**

\[
2695 \text{ mm}^3 = \frac{2695 \text{ mm}^3}{1} \times \frac{(1 \text{ m})^3}{(10^3 \text{ mm})^3} \times \frac{(10^2 \text{ cm})^3}{(1 \text{ m})^3} \]

\[
= \frac{2695 \text{ mm}^3}{1} \times \frac{1 \text{ m}^3}{10^3 \text{ mm}^3} \times \frac{10^6 \text{ cm}^3}{1 \text{ m}^3} = 2.695 \text{ cm}^3
\]

Notice that the decimal point is just shifted three places to the left since we are dividing by $10^3$.

**Alternate Solution:** If you take a look at a ruler, you will notice that there are 10 mm in each centimeter. Another method for solving this problem is to use this fact.

\[
2695 \text{ mm}^3 = \frac{2695 \text{ mm}^3}{1} \times \frac{1 \text{ cm}^3}{(10 \text{ mm})^3} = \frac{2695 \text{ mm}^3}{1} \times \frac{1 \text{ cm}^3}{10^3 \text{ cm}^3}
\]

\[
= \frac{2695 \text{ mm}^3}{1} \times \frac{1 \text{ cm}^3}{1000 \text{ mm}^3} = 2.695 \text{ cm}^3
\]

As seen from both calculations 2695 mm$^3$ is equal to 2.695 cm$^3$.

In some volume problems involving metric units, you may need to convert from cubic units to liters. The key here is to remember that the metric system was created so that 1 mL = 1 cm$^3$. The key to doing conversions between cubic metric units and liters is to use a unit fraction involving this conversion factor.

**Example:** Convert 187 mm$^3$ to mL.

**Solution:**

\[
187 \text{ mm}^3 = \frac{187 \text{ mm}^3}{1} \times \frac{(1 \text{ m})^3}{(10^3 \text{ mm})^3} \times \frac{(10^2 \text{ cm})^3}{(1 \text{ m})^3} \times \frac{1 \text{ mL}}{1 \text{ cm}^3}
\]

\[
= \frac{187 \text{ mm}^3}{1} \times \frac{1 \text{ m}^3}{10^3 \text{ mm}^3} \times \frac{10^6 \text{ cm}^3}{1 \text{ m}^3} \times \frac{1 \text{ mL}}{1 \text{ cm}^3} = 0.187 \text{ mL}
\]

Notice that we simply shifted the decimal point three places to the left since we were dividing by $10^3$. Thus 187 cubic millimeters is the same as 0.187 milliliters.

Let's turn to an application now.

**Example:** Determine the volume of a box that has dimensions 9 mm by 7.5 mm by 27 mm in cubic centimeters.
Solution: As with the area problem discussed earlier, there are two ways to do this problem. We can convert each of the dimensions to centimeters and then calculate the volume, or we can calculate the volume and then convert the volume to cubic centimeters.

**Solving by Converting the Dimensions:**
To convert millimeters to centimeters we simply have to shift the decimal point one place to the left since we are multiplying each measurement by \( \frac{1 \text{ cm}}{10 \text{ mm}} \) (since there are 10 mm in 1 cm as mentioned previously). This means the dimension of the box is 0.9 cm by 0.75 cm by 2.7 cm. Volume of a box is found by taking the length times the width times the height.

\[
V = (\text{length})(\text{width})(\text{height}) = (0.9 \text{ cm})(0.75 \text{ cm})(2.7 \text{ cm}) = 1.8225 \text{ cm}^3
\]

A box with dimensions of 9 mm by 7.5 mm by 27 mm has a volume of 1.8225 cm\(^3\) or about 1.8 cm\(^3\).

**Solving by Calculating the Volume First:**
As mentioned above, the volume of a box is found by taking the length times the width times the height as follows:

\[
V = (\text{length})(\text{width})(\text{height}) = (9 \text{ mm})(7.5 \text{ mm})(27 \text{ mm}) = 1822.5 \text{ mm}^3
\]

After calculating the volume we do the conversion being careful to cube the linear conversion factor.

\[
1822.5 \text{ mm}^3 \times \left( \frac{1 \text{ m}^3}{10^3 \text{ mm}^3} \right) \times \left( \frac{10^2 \text{ cm}^3}{1 \text{ m}^3} \right) = 1.8225 \text{ cm}^3
\]

Consistent with the other method, we see that the volume of the box is 1.8225 cm\(^3\) or about 1.8 cm\(^3\). Note: We could also do this conversion by using the fact that 10 mm is equal to 1 cm and thus \( (1 \text{ cm})^3 = (10 \text{ mm})^3 \).

**Your Turn!!**

1. Convert 9205 in\(^3\) to ft\(^3\).
   1) _________________________________

2. Convert 0.038 ft\(^3\) to in\(^3\).
   2) _________________________________

3. Convert 427 ft\(^3\) to yd\(^3\).
   3) _________________________________

4. Convert 0.78 yd\(^3\) to ft\(^3\).
   4) _________________________________
5. Convert 8.75 gal to ft$^3$.  

6. Convert 3.75 ft$^3$ to gallons.  

7. Convert 326 L to m$^3$.  

8. Convert 3260 mm$^3$ to liters.  

9. Convert 3.5 m$^3$ to liters.  

10. Convert 56 mL to mm$^3$.  

11. Convert 0.0575 m$^3$ to cm$^3$.  

12. Convert 2345 mm$^3$ to cm$^3$.  

13. Find the volume of a box in cubic yards that has dimensions 8 ft 3 in by 7 ft 9 in by 3 ft 6 in.  

14. Find the volume of a box in cubic centimeters that has dimensions 11 mm by 5.8 mm by 25 mm.  


Section 4.4: Conversion Between Metric and English Units

<table>
<thead>
<tr>
<th>Linear Measure:</th>
<th>Area Measure:</th>
<th>Volume Measure:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 inch = 2.54 centimeters</td>
<td>1 in² = 6.451 6 cm²</td>
<td>1 oz = 29.573 530 mL</td>
</tr>
<tr>
<td>1 meter = 3.280 840 feet</td>
<td>1 m² = 1.195 990 yd²</td>
<td>1 L = 1.056 688 qt</td>
</tr>
<tr>
<td>1 meter = 1.093 613 yard</td>
<td>1 mi² = 2.589 988 km²</td>
<td>1 gal = 3.785 412 L</td>
</tr>
<tr>
<td>1 mile = 1.609 344 kilometer</td>
<td>1 acre = 40.468 5642 are</td>
<td>1 ft³ = 7.480 519 gal</td>
</tr>
<tr>
<td></td>
<td>1 are = 100 square meters</td>
<td>1 m³ = 28.316 847 L</td>
</tr>
<tr>
<td></td>
<td>1 hectare = 100 are = 2.471 054 acre</td>
<td>1 L = 61.023 744 in³</td>
</tr>
<tr>
<td>Weight:</td>
<td>Speed:</td>
<td>Temperature:</td>
</tr>
<tr>
<td>1 kilogram = 2.204 623 pounds</td>
<td>1 m / s = 2.236 936 mph</td>
<td>°F = \frac{9°F}{5°C} × Temp°C + 32°F</td>
</tr>
<tr>
<td>1 oz = 28.349 523 g</td>
<td>1 m / s = 3.280 840 ft / s</td>
<td>°C = \frac{5°C}{9°F} × (Temp°F − 32°F)</td>
</tr>
<tr>
<td></td>
<td>1 mph = 1.609 344 kph</td>
<td>°K = \frac{1°K}{1°C} × Temp°C + 273.15°K</td>
</tr>
</tbody>
</table>

In the United States we often use the English System rather than the Metric System. However, we import and export products to countries that use the Metric System instead. This makes it important for us to be able to convert measurements between the two systems. Conversions between metric and English units require conversion factors. Because the systems are completely different, these conversions will involve decimal values. Depending on the level of accuracy required for our calculation we can use conversion factors that go to different numbers of decimal places. At the end of this chapter there is an appendix that lists conversion factors between the systems. Many of these will list eight decimal places of accuracy. If our calculations do not require this level of accuracy, we will often use a conversion factor that is rounded off to smaller number of decimal places.

How do we convert between units in the two systems? Just as we did when converting units within a system, we will use unit fractions to convert between units in the two systems.

Example:  \textit{Convert 5.6 miles to kilometers.}

Solution: \[\frac{5.6 \text{ miles}}{1} \times \frac{1 \text{.609344 km}}{1 \text{ mile}} = \frac{5.6 \times 1.609344}{1} \text{ km} \]

\[= 9.0123264 \text{ km} \approx 9.0 \text{ km} \]

5.6 miles is about 9.0 kilometers.

Example:  \textit{Convert 5.4 cm² to in².}

Solution: As previously, we must be careful as we convert square units. In order to obtain the correct conversion factor, we must square the linear conversion factors.
\[
5.4 \text{ cm}^2 = \frac{5.4 \text{ cm}^2}{1} \times \frac{(1 \text{ in})^2}{(2.54 \text{ cm})^2} = \frac{5.4 \text{ cm}^2}{1} \times \frac{1 \text{ in}^2}{6.4516 \text{ cm}^2}
\]
\[
= \frac{5.4}{6.4516} \text{ in}^2 = 0.837001674 \text{ in}^2 \approx 0.84 \text{ in}^2
\]
Thus 5.4 cm\(^2\) is about 0.84 in\(^2\).

**Example:** Convert 1.80 gallons per minute to \(m^3\) per hour.

**Solution:**
\[
1.80 \text{ gal/min} = \frac{1.80 \text{ gal}}{1 \text{ min}} \times \frac{3.7854 \text{ L}}{1 \text{ gal}} \times \frac{10^{-3} \text{ m}^3}{1 \text{ L}} \times \frac{60 \text{ min}}{1 \text{ hour}}
\]
\[
= \frac{1.80 \times 3.7854 \times 10^{-3} \times 60 \text{ m}^3}{1 \text{ L}} = 0.4088232 \text{ m}^3/\text{hr} \approx 0.409 \text{ m}^3/\text{hr}
\]
Thus 1.80 gallons per minute is about 0.409 cubic meters per hour.

The conversion between temperatures in the two systems requires the use of a formula rather than the use of unit fractions and conversion factors. To convert from a centigrade temperature to a Fahrenheit temperature, we use the formula:
\[
\text{Temp}^\circ F = \text{Temp}^\circ C \times \frac{9}{5} + 32^\circ F
\]

**Example:** Convert 40°C to degrees Fahrenheit.

**Solution:**
\[
\text{Temp}^\circ F = 40^\circ C \times \frac{9}{5} + 32^\circ F = (72 + 32) ^\circ F = 104^\circ F
\]
To convert from a Fahrenheit temperature to a centigrade temperature, we use the formula
\[
\text{Temp}^\circ C = (\text{Temp}^\circ F - 32) \times \frac{5}{9}
\]

**Example:** Convert 10°F to degrees Celsius.

**Solution:**
\[
\text{Temp}^\circ C = (10 - 32) \times \frac{5}{9} = \frac{-22 \times 5}{9} = \frac{-110}{9} = \frac{-122}{9} \approx -12.2
\]
Thus 10°F is equivalent to about \(-12.2^\circ C\).

Temperature is a measure of heat. As a result, some scientists do not find having negative temperatures meaningful in their work. A third scale of measuring temperature, measured in degrees Kelvin, is built using the centigrade scale but it eliminates negative temperatures. In this system 0 degrees is at absolute zero. Absolute zero is the coldest possible temperature; the temperature at which molecules stop moving. To convert from degrees Celsius to degrees Kelvin we simply use the formula: \(\text{Temp}^\circ K = \text{Temp}^\circ C + 273.15\).

**Example:** Convert 12.5°C to degrees Kelvin.

**Solution:**
\[
\text{Temp}^\circ K = 12.5 + 273.15 = 285.65
\]
Thus 12.5°C is the same as 285.65°K
To convert from degrees Kelvin to degrees Celsius we simply use the formula: $\text{Temp}^{\circ}\text{C} = \text{Temp}^{\circ}\text{F} - 273.15$

**Example:**  Convert $25.8^{\circ}K$ to degrees Celsius.

**Solution:**  

$\text{Temp}^{\circ}\text{C} = 25.8 - 273.15 = -247.35$

Thus $25.8^{\circ}K$ is the same as $-247.35^{\circ}C$.

**Your Turn!!**

1. Convert 7 ft to meters.  

2. Convert 9 inches to centimeters.

3. Convert 14 meters to yards.

4. Convert 26.5 yards to meters.

5. Convert 15 km to miles.

6. Convert 82 miles to km.

7. Convert 25 m to feet.

8. Convert 17.5 cm to inches.


10. Convert 280 L to gallons.

11. Convert 23 qt to liters.

12. Convert 19 liters to quarts.

13. Convert 2.49 gallons to milliliters.
14. Convert 4.5 liters to ounces.  
14) ________________________________

15. Convert 32 lb to kg.  
15) ________________________________

16. Convert 7 oz to grams.  
16) ________________________________

17. Convert 16 kg to lb.  
17) ________________________________

18. Convert 126 g to ounces.  
18) ________________________________

19. Convert 55 km/hr to mi/hr.  
19) ________________________________

20. Convert 62 mi/hr to km/hr.  
20) ________________________________

21. Convert 83 mph (meters per hour) to ft/min.  
21) ________________________________

22. Convert 2.56 square feet to $cm^2$.  
22) ________________________________

23. Convert 179 cm$^3$ to in$^3$.  
23) ________________________________

24. Convert 465 ft$^2$ to m$^2$.  
24) ________________________________

25. Convert 3.2 m$^3$ to ft$^3$.  
25) ________________________________

26) ________________________________

27. Convert 25.8 mi/gal to km/L.  
27) ________________________________

28. Convert 35°C to °F.  
28) ________________________________

29. Convert 98°F to °C.  
29) ________________________________

30. Convert -23.5°F to °C.  
30) ________________________________
31. Convert \(-5.75^\circ C\) to \(^\circ F\).  

32. Convert \(47.5^\circ K\) to \(^\circ C\).  

33. Convert \(-125.5^\circ C\) to \(^\circ K\).  

34. Convert \(-17.5^\circ C\) to \(^\circ K\).  

35. Convert \(365^\circ K\) to \(^\circ C\).  

36. Convert \(-36.5^\circ F\) to \(^\circ K\).  

37. Convert \(46^\circ F\) to \(^\circ K\).  

38. Convert \(65.4^\circ K\) to \(^\circ F\).  

39. Convert \(301.6^\circ K\) to \(^\circ F\).  

40. A wire that is 13 mm wide is how many inches wide?  

41. The Weston’s left Seattle on a cruise and where told that they would travel 76 miles before they entered Canadian water. At that point they would travel an additional 157 km until their final destination. How many kilometers was the entire trip?  

42. Ralph got a deal on a car that he bought used in the Bahamas. The only problem was the speedometer was marked only in kilometers per hour (km/hr). While driving through the state of Texas he was pulled over for traveling too slowly in the express lane of an interstate. He told the officer that he was doing 65, but this was km/hr. How fast was he going in MPH, miles per hour?  

43. How large would a farm of 320 acres be if it were measured in square kilometers?
44. A box has dimensions of 5 ft 6 in by 3 ft 9 in by 3 ft 4 in. How many cubic meters is this?
   44) ________________________________

45. A car has a gas tank with a capacity of 50 L. If the car gets 33.5 miles per gallon, how many km can the
car travel on a full tank?
   45) ________________________________

Chapter 4 Practice Exam

1. Perform the following calculation with measurement numbers: 15 yd. 1ft. – 9 yd. 2ft.
2. Perform the following calculation with measurement numbers: 1.6 MV + 14.4 MV
3. Perform the following calculation with measurement numbers: 4.75 in² × 3.6 in
4. Perform the following calculation with measurement numbers: 425 km ÷ 95 km per hour
5. Convert 56°C to °F.
6. Convert 14.5 liters to gallons.
7. Convert 18.5 lbs. to kg.
8. Convert 976 ft³ to in³.
9. Convert 36.5 mA to A.
10. Convert 0.945 m to cm.
11. Convert 75 m/sec to km/h.
12. Convert 54.5 yd² to in².
13. Convert 21.65 ft³ to gal.
15. Convert 26.5 in³ to cm³.
16. What size bolt, to the nearest 64th of an inch, will fit a hole 0.365" in diameter?
17. Joe’s Service station has a drum that has 40 gallons and 3 quarts of oil left in it. How many oil changes
can they perform if each oil change requires 6 quarts of oil?
18. To pour a concrete driveway you must order the concrete in cubic yards. If the driveway is to be 10 feet wide, 75 feet long and 4 inches thick, how many cubic yards of concrete are needed? Round your answer to one decimal place.

19. A 2.95 meter long sash cord has become frayed at both ends, so 1.25 cm is trimmed from each end. How long is the remaining cord?

20. A car has a gas tank with a capacity of 46 L. If the car gets 54.5 kilometers per gallon, how many miles can the car travel on a full tank?
Chapter 5: Geometry

Geometry is a word from Greek which combines the word geo meaning Earth in Greek and the word metry which means to measure. So geometry literally means measuring the Earth. More generally geometry is the study of shapes, their properties, and how to measure them.

Section 5.1: Plane geometry

Basic Facts About Points and Lines:

- Two different points define one and only one line that passes through both of them.
- We shall use capital letters such as A to label points and a pair of letters such as AB to label the segment of the line between A and B.
- Line segments are measured in units of length such as feet or meters.
- The shortest distance between the two points is along the line passing through them.
- Two lines in a plane either meet (or “intersect”) at a single point or are parallel, meaning that the two lines never touch.
- Where the lines meet four “openings” or angles are formed as shown below. The symbol \( \angle \) is used for the word angle. (See figure below)

\[
\begin{align*}
\triangle CAE & \quad \triangle 1 \\
\square CAB & \quad \triangle BAD
\end{align*}
\]

Angles

The vertex of an angle is the common point shared by the two line segments or lines that form the sides of the angle.

One way to name angles is to write the “angle symbol” \( \angle \), then the name for a point on one side, the name for the vertex, and then the name of a point on the other side.

Thus \( \angle BAD \) is the angle with vertex at A and with sides that include segments AB and AD. This same angle could also be labeled as \( \angle DAB \).

Another way is to write the angle symbol, and then a single letter or number, like \( \angle 1 \) in the above diagram.
Measuring Angles
- The most common unit for measuring angles is called the **degree**.
- The symbol for the degree is: °
- The Babylonians decided that there are 360 degrees (360°) in one full rotation.
- A **right angle** is one-fourth of a full rotation and measures 90°.
- A **straight angle** is one-half of a full rotation and measures 180°.
- An angle between 0° and 90° is called an **acute angle**.
- An angle between 90° and 180° is called an **obtuse angle**.

Complementary and Supplementary Angles
- Since there are 360° in a circle, the angular measure of a straight line is 180°.
- In a square or right angle there are 90°.
- **Complementary angles** have measures that add up to 90°.
- **Supplementary angles** have measures that add up to 180°.

Fractions of a Degree
- An old-school technique for talking about angle measures less than a degree is to subdivide the degree in the same way we subdivide time: into minutes and seconds.
- One minute of angle is one-sixtieth of a degree: 1′ = \(\frac{1^\circ}{60}\) \(\Leftrightarrow\) 60′ = 1°
- One second of angle is one-sixtieth of a minute: 1″ = \(\frac{1′}{60}\) \(\Leftrightarrow\) 60″ = 1′
- Example of an angle measurement stated in degrees, minutes, and seconds (DMS): 32°24′17″

Converting from degrees, minutes, and seconds (DMS) to decimal degrees (DD)
- Example: convert 32°24′17″ to DD
- Divide the number of second by sixty to convert it to an equivalent number in minutes.
  - 17″ \(\left(\frac{1′}{60}\right)\) ≈ 0.283 333′
- Add that to the number of minutes, then divide by sixty again to convert to an equivalent number in degrees.
  - 24.283 333′ \(\left(\frac{1^\circ}{60}\right)\) ≈ 0.404 722°
- Add that number to the number of degrees, and round to an appropriate number of places:
  - 32°24′17″ ≈ 32.405°
- Note that 1″ = \(\frac{1^\circ}{3600}\) ≈ 0.000 278°. Thus, stating DMS measurements to the nearest thousandth of a degree won’t result in much round-off error, but stating your answer beyond a ten-thousandth of a degree implies a precision not conveyed by mere seconds.
- Note: on a graphing calculator, you can type in a DMS measurement using the degree and minute symbols found in the ANGLE menu, and the quotation marks for the seconds.
Converting from decimal degrees (DD) to degrees, minutes, and seconds (DMS)

- Example: convert 185.659° to DMS
- Multiply the decimal portion by sixty to convert it to an equivalent number in minutes.
  \[
  0.6592\left(\frac{60'}{1°}\right) = 39.552'
  \]
- Multiply the resulting decimal portion by sixty again to convert to an equivalent number in seconds.
  \[
  0.552\left(\frac{60''}{1'}\right) = 33.12''
  \]
- Round to an appropriate number of places:
  \[
  185.659° \approx 185°39'33''
  \]
- Note: on a graphing calculator, you can type in a DD measurement and convert it to a DMS measurement using the \( \rightarrow \) DMS function found in the ANGLE menu.

When two lines intersect, the angles that are opposite each other are equal. These equal angles formed by two intersecting lines are called **vertex (or vertical) angles**.

![Diagram showing vertex angles](image)

**Parallel lines** lie in the same plane and do not intersect. A **transversal** is a line that intersects two parallel lines.

- Different angles formed by the transversal are given special names to reflect special relationships between their measures:
  - Corresponding angles are on the same side of the transversal and on the same corresponding sides of the parallel lines. Corresponding angles have equal measures.
  - Alternate interior angles have equal measures.
  - Alternate exterior angles have equal measures.
  - Interior angles on the same side of the transversal are supplementary (add to 180°).

Consider the following diagram:

\[
\triangle d = 144°39'35''
\]

**SOLUTION:** Once the measure of \( \angle d \) is known, the remaining three can be determined. Since they are vertex angles, \( \angle b = \angle d = 144°39'35'' \). Since \( \angle a \) is the supplement to \( \angle d \), \( \angle a = 180° - 144°39'35'' \). So
144°39'35" = 179°59'60" – 144°39'35". So 180° - 144°39'35" = 35°20'25" = \angle c. This same calculation can also be done on a on most scientific calculators using a few keystrokes.

Consider a pair of parallel lines crossed by a third line (called the transversal) as shown below. If we imagine that point B is superimposed, or in other words, moved so that it is directly over point F by moving segment CD onto the line through EG, the corresponding angles \( \angle EFB \) and \( \angle CBA \) are equal. Similarly, \( \angle EFH = \angle CBF \), \( \angle ABD = \angle BFG \), and \( \angle FBD = \angle HFG \). From the equality of vertex angles, the alternating interior (alternate sides of the transversal, inside the two parallel lines) are equal, i.e., \( \angle EFB = \angle FBD \) and \( \angle CBF = \angle BFG \). Similarly, the alternating exterior (alternate sides of the transversal, outside the two parallel lines) are equal, i.e., \( \angle EFH = \angle ABD \) and \( \angle CBA = \angle HFG \).
Polygons
Polygons are closed figures in the plane whose sides are line segments.

Triangles
The simplest polygon is the three-sided triangle. The points at the corners A, B, and C are the vertices of the triangle, and the angles ∠BAC, ∠BCA, and ∠ABC are called the interior angles of the triangle. The triangle is often then labeled as triangle ABC.

Important Fact:
The interior angles of a triangle always add up to 180°.

Types of Triangles
Triangles are classified according to their angles and sides.

- Angle classifications:
  - Acute triangles have all angles less than 90°.
  - Right triangles have one angle of 90°.
  - Obtuse triangles have one angle greater than 90°.

- Side classifications:
  - Equilateral triangles have all sides the same length. Also, all angles are 60°.
  - Isosceles triangles have two sides of the same length, and the angles opposite those two sides are equal.
  - Scalene triangles have no sides of the same length.

A triangle is called isosceles if two sides are equal. Consider the isosceles triangle ACB with AC = CB; from C construct the segment CD to D the mid point (i.e., AD = DB) of AB. Now by SSS triangle ADC is congruent to triangle BDC. Thus, ∠ADC = ∠BDC, and since these two angles sum to 180°, CD is perpendicular (makes a right angle) to AB. This is indicated in the diagram by the “little” box at D. Also ∠CAB = ∠CBA; in words, the angles opposite to the equal sides are also equal.
Consider the following isosceles triangle with $\angle 1$ unspecified. Since the angles opposite the equal sides must be equal, $\angle 1 = 180^\circ - 2(39^\circ) = 180^\circ - 78^\circ = 102^\circ$.

![Isosceles Triangle Diagram]

A triangle is called **equilateral** if all three sides are equal.

![Equilateral Triangle Diagram]

Of course, equilateral triangles are also isosceles, so $AB = BC$ and the opposite angles $\angle BAC$ and $\angle BCA$ are equal. But $AC = BC$, so $\angle ABC$ and $\angle BAC$ are equal. Thus all three angles are equal and since they sum to $180^\circ$, we have that for an equilateral triangle all of the internal angles equal $60^\circ$.

**EXAMPLE:** In the triangle below the missing angle $\angle 1$ is calculated as follows:

**SOLUTION:** $\angle 1 = 180^\circ - 97^\circ 13' - 30^\circ 59' = 51^\circ 48'$
**Congruent triangles** have all the same side lengths and all the same angles. Congruent is the proper way to say that two triangles are “equal” or “the same”. We use the word congruent because there are really six measurements that have to be equal for two triangles to be the same: the three angle measurements, and the three side length measurements.

The three rules for congruency when using triangles:
- **Side Angle Side (abbreviated SAS)**,
- **Angle Side Angle (abbreviated ASA)**, and
- **Side Side Side (abbreviated SSS)**.

**Rules of Congruence**

1. **SAS**

   ![Diagram of SAS congruence](image)

2. **ASA**

   ![Diagram of ASA congruence](image)

3. **SSS**

   ![Diagram of SSS congruence](image)

A **right triangle** is a triangle with a $90^\circ$ interior angle. The large side opposite the $90^\circ$ angle is called the **hypotenuse** while the remaining two sides are called **legs**. In the diagram shown, the hypotenuse is $c$ and the legs are $a$ and $b$. The two **acute** (less than $90^\circ$) angles are $\angle 1$ and $\angle 2$ with $\angle 1$ opposite to the side of length $a$ and $\angle 2$ opposite to the
The Pythagorean Theorem: \( c = \sqrt{a^2 + b^2} \).

To calculate a leg, say \( a \), knowing the hypotenuse and the other leg, \( b \), rearrange the formula to

\[
a = \sqrt{c^2 - b^2} \quad \text{or} \quad \text{leg} = \sqrt{\text{hypotenuse}^2 - \text{other leg}^2}.
\]

**Remember** the square root symbol is also a grouping symbol. Parantheses need to be used!

**EXAMPLE:** Find the hypotenuse of the right triangle pictured below.

To find the missing hypotenuse of the following right triangle, we compute as follows:
\[ c = \sqrt{(8.3\text{m})^2 + (10.6\text{m})^2} = \sqrt{68.89\text{m}^2 + 112.36\text{m}^2} = \sqrt{181.25\text{m}^2} = 13.5\text{ m} \]

**Note:** The implied parenthesis inside the square root has been made explicit as required to get the correct answer on the calculator. The units work out to be linear units as required for a length since \( \sqrt{m^2} = m \).

**EXAMPLE:** consider finding the unknown quantities in the following right triangle. Find the length of the missing leg \( x \).

**SOLUTION:** The missing dimension \( x \) is a leg and can be calculated from

\[ x = \sqrt{(16\text{m})^2 - (12\text{m})^2} = \sqrt{112\text{m}^2} = 10.6\text{ m} \]

The missing angle is complementary to \( 41^\circ24'35'' \), so

\[ \angle 1 = 90^\circ - 41^\circ24'35'' = 48^\circ35'25'' \]

**Other Polygons** (Polygons with more than 3 sides)

- In general, if a polygon has \( n \) sides, then the sum of the interior angles is \( (n - 2)(180^\circ) \).
Consider the five-sided pentagon above. By drawing three triangles from one of the vertices, we see that the sum of the internal angles of the pentagon is the sum of all the internal angles in the three triangles or $3(180^\circ) = 540^\circ$. A similar argument for an n-sided polygon shows that the sum of the internal angles is $(n-2)(180^\circ)$.

**EXAMPLE:**

What do you get when you add up all of the interior angles in a 7 sided polygon?

**SOLUTION:** In this case $n = 7$, so $(n-2)(180^\circ) = (7-2)(180^\circ) = 5(180^\circ) = 900^\circ$.

**EXAMPLE:**

What do you get when you add up all of the interior angles in a 100 sided polygon?

**SOLUTION:** In this case $n = 100$, so $(n-2)(180^\circ) = (100-2)(180^\circ) = 98(180^\circ) = 17,640^\circ$.

**Perimeter and Area**

**Perimeter** is the total linear distance around the boundary of a polygon (or any closed shape). **Area** is the number of unit squares of measurement it takes to fill a closed shape.

**Example:** the perimeter of the pentagon shown below is computed as $P = 3\text{ in} + 6\text{ in} + 5\text{ in} + 3\text{ in} + 5\text{ in} = 22\text{ in}$.

![Pentagon Diagram](image)

**Area and Perimeter of a Rectangle**

Consider a rectangle of width $W$ and length $L$. The perimeter is computed as $P = W + W + L + L = 2W + 2L$. 

![Rectangle Diagram](image)
The area of a rectangle, which measures the amount of “two dimensional space” inside the rectangle is given by \( A = W \cdot L \).

**Note**: perimeter always has units of length, while area always has units of length\(^2\).

For the following rectangle, we compute \( P = 2(1.85 \text{ cm}) + 2(2.20 \text{ cm}) = 8.10 \text{ cm} \).

![Rectangle](image)

The area is given by \( A = 1.85 \text{ cm} \times 2.20 \text{ cm} = 4.07 \text{ cm}^2 \).

**Area and Perimeter of a Parallelogram**

A parallelogram is a four sided polygon (or quadrilateral) with opposite sides parallel. All rectangles are parallelograms, but in a generic parallelogram the angle between adjacent sides is not necessarily \( 90^\circ \).

Consider the parallelogram shown below with base \( b \) and perpendicular distance \( h \) between the top and bottom sides. Imagine cutting off a right triangle from the left end and moving it to the right end. Since the left and right sides are parallel, this right triangle fits perfectly to make a rectangle of dimensions \( b \) and \( h \). So we have for a parallelogram that \( A = b \cdot h \).
**Area and Perimeter of a Triangle**

Next consider a generic triangle with bottom side (base) of length $b$ and perpendicular distance (height) $h$ from the base to the top vertex.

Imagine making an exact copy of this triangle and joining it to the original triangle as shown.

The result is a parallelogram of base $b$ and height $h$. Since the area of the original triangle is half of the area of this parallelogram, we arrive at the result that the area of a triangle is given by

$$A = \frac{1}{2}bh.$$  

A more detailed argument shows that for a triangle with sides $a$, $b$, and $c$ the area can be calculated from Heron’s formula:

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

with the semi-perimeter $S$ given by the formula: $S = \frac{a+b+c}{2}$. 
EXAMPLE:

Consider calculating the perimeter and area of the following triangle.

\[
\begin{align*}
th \quad 15.6 \text{ in} \\
16.5 \text{ in} \\
15.6 \text{ in}
\end{align*}
\]

The perimeter is just the sum of the lengths of the three sides.

\[
P = 15.6 \text{ in} + 15.6 \text{ in} + 16.5 \text{ in} = 47.7 \text{ in}
\]

To calculate the area, we could calculate the height by dropping a perpendicular from the top vertex. Since the triangle is isosceles, this bisects the 16.5 in base. Using the Pythagorean Theorem we compute \( h \) as follows:

\[
h = \sqrt{(15.6 \text{ in})^2 - (16.5 \text{ in}/2)^2} = \sqrt{175.30 \text{ in}^2} = 13.2 \text{ in}
\]

Then the area is calculated as half the base times the height.

\[
A = 0.5 \times 16.5 \text{ in} \times 13.2 \text{ in} = 109 \text{ in}^2
\]

Another way to calculate the area is to use Heron’s formula.

\[
S = \frac{47.7 \text{ in}}{2} = 23.85 \text{ in}
\]

\[
A = \sqrt{23.85 \text{ in}(23.85 \text{ in} - 15.6 \text{ in})(23.85 \text{ in} - 15.6 \text{ in})(23.85 \text{ in} - 16.5 \text{ in})} = \sqrt{11931 \text{ in}^4} = 109 \text{ in}^2
\]

Area and Perimeter of a Trapezoid

A quadrilateral with two opposite sides parallel is called a trapezoid. Suppose that the two parallel faces have lengths \( a \) and \( b \) and are separated by a perpendicular distance \( h \).
Imagine making an exact copy of this trapezoid and joining it to the original trapezoid as shown.

The result is a parallelogram of base $a + b$ and height $h$. Since the area of the original trapezoid is half of the area of this parallelogram, we arrive at the result that the area of a trapezoid is given by the following formula:

$$A = \frac{1}{2}(a + b) \cdot h$$

**EXAMPLE:**

As an application consider calculating the perimeter and area of the following trapezoid.

The first step is to calculate the length $x$. Using the Pythagorean Theorem the length of the base of the right triangles that form the sides of the trapezoid is computed as follows:

$$\sqrt{(8 \text{ cm})^2 - (6 \text{ cm})^2} = \sqrt{28 \text{ cm}^2} = 5.29 \text{ cm}.$$
Then $x = 8\, \text{cm} + 2 \times 5.29\, \text{cm} = 18.58\, \text{cm}$. The perimeter is just the sum of the lengths of the four sides. 

$P = 18.6\, \text{cm} + 8\, \text{cm} + 8\, \text{cm} + 8\, \text{cm} = 42.6\, \text{cm}$.

To calculate the area, we could add the area of the two right triangles that form the sides of the trapezoid to the area of the 8 cm by 6 cm central rectangle.

$$A = 2 \times \frac{1}{2} \times 5.29\, \text{cm} \times 6\, \text{cm} + 8\, \text{cm} \times 6\, \text{cm} = 79.7\, \text{cm}^2$$

We get the same result by using the formula for the area of a trapezoid.

$$A = \frac{1}{2} \times (18.58\, \text{cm} + 8\, \text{cm}) \times 6\, \text{cm} = 79.7\, \text{cm}^2$$

**Circles**

![Diagram of a circle with radius r, diameter D, and center C]

A circle is formed by generating all points in a plane which are a fixed distance called the radius from a center point here labeled as C. A line segment with end points on the circle that passes through the center is called a diameter. D and r symbolize the lengths of the diameter and radius, respectively.

Since $AC = CB = r$ and $D = AB = AC + CB = 2r$, we have the following formulas:

$$D = 2r \quad \text{and} \quad r = \frac{D}{2}$$

All circles are similar. By this we mean that all circles have the same shape. The distance around the boundary of a circle is called its circumference which is like the perimeter of a polygon.

Imagine that we have two circles. The first labeled as 1 has radius $r_1$, diameter $D_1$ and circumference $C_1$. The second labeled as 2 has radius $r_2$, diameter $D_2$ and circumference $C_2$. Since all circles are scale models of each, the ratio of circumference to diameter is the same for both circles. The value of this ratio is symbolized by the Greek letter lower case pi. Pi is symbolized by $\pi \approx 3.141592654$. The value of pi is slightly larger than 3.
fact the decimal approximation to pi is known to over a billion digits! There may be a pi button on your calculator. Pi is approximately equal to 3.14159. However 3.14 will be good enough in most cases.

\[ C = \pi D \]

The “perimeter” or a circle is technically called the **circumference** of the circle.

**Area of a Circle**

\[ A = \pi r^2 \]

**EXAMPLE:**

The circumference and area of a circle of diameter 2.500 in are computed by the following calculations:

\[ C = \pi D = \pi \times 2.500 \text{ in} = 7.854 \text{ in} \]

\[ r = \frac{D}{2} = \frac{2.500 \text{ in}}{2} = 1.250 \text{ in} \]

\[ A = \pi r^2 = \pi \times (1.250 \text{ in})^2 = 4.909 \text{ in}^2 \]

**Your Turn!!**

If \( \angle 1 = 85^\circ 19'56'' \), what is the measure of \( \angle 2 \)?

1. \( \angle 2 = \underline{\text{___________}} \)
If line $L_1$ is parallel to line $L_2$ and $\angle CAB = 76^\circ 59'$ and $\angle CBA = 46^\circ 29'$ and $\angle CEG = 61^\circ 29'$, find the requested missing $\angle$'s.

2. $\angle ACB =$ ____________

3. $\angle CFH =$ ____________

4. $\angle CGE =$ ____________

If BE is parallel to AD and $\angle ACB = 64^\circ 17'$ and $\angle BAC = 66^\circ 35'$, find the missing $\angle$'s.

5. $\angle ABC =$ ____________

6. $\angle BCD =$ ____________

7. $\angle CBE =$ ____________

Find the measure of the missing angles.

8. $\angle 2 =$ ____________

9. $\angle 3 =$ ____________
10. $\angle 1 =$
11. $\angle 2 =$
12. $\angle 3 =$
13. $\angle 1 =$

For each figure below calculate the distance around (perimeter) $P$.

14. $P =$
15. $P =$

For each figure below calculate both the area, $A$, and the distance around (perimeter or circumference), $P$ or $C$.
16. \( P = \) __________
17. \( A = \) __________
18. \( P = \) __________
19. \( A = \) __________
20. \( P = \) __________
21. \( A = \) __________
22. \( C = \) __________
23. \( A = \) __________
For each figure below calculate the requested missing information.

24. \( \angle 1 = \) __________

25. \( a = \) ____________

1. \( \angle 1 = \) ____________

2. \( a = \) ____________

3. \( P = \) ____________

29. \( A = \) ____________
How many square feet of flooring does the following room have?

The cross section of a shed is shown below. Determine the height $h$ above the ground and $A$, the total area of the building’s cross section.

A 2.25 cm diameter hole is drilled in a 4.5 cm diameter circle. What area is left after the hole has been drilled?
**Section 5.2: Radian Measure and its Applications**

In addition to DMS (Degrees, Minutes, and Seconds) and DD (Decimal Degrees), there is another system for measuring angles called **radian measure**. The basis of this system is that all sectors of a circle having the same central angle \( \theta \) (theta) are similar. Thus, for a given central angle, the ratio of the arc length of the sector to the radius of the circle is a constant. We define this constant as the radian measure of the angle. We say that the arc length is “cut-off” or subtended by the central angle theta at a given radius \( r \).

\[
\theta \text{(measured in radians)} = \frac{s}{r} = \frac{S}{R}
\]

Arc length = radius \( r \) \times central angle in radians

\[s = r \cdot \theta \quad (\theta \text{ measured in radians})\]

\( C \pi r \) \( 2 \ r = \) \( \pi \) (full revolution angle in radians)

so a full revolution of \( 360^\circ = 2\pi \) radians

or more simply \( \pi \) rad = \( 180^\circ \)

To convert decimal degree measurements to radians we use the conversion factor \( 1 = \frac{\pi \text{ radians}}{180^\circ} \).

To convert radian measure to decimal degrees we use the conversion factor \( 1 = \frac{180^\circ}{\pi \text{ radians}} \).

The next three examples use these principles.

**Example:** A car on a circular track of radius 0.4 miles is travelling 125 mph. In 2 seconds through what central angle measured from the center of the track does the car travel?

**Solution:** The arc length subtended by the central angle is just the distance travelled by the car. Since distance is speed multiplied by time, we have

\[
S = \left( \frac{125 \text{ miles}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \right) \cdot 2 \text{ sec} = .0694 \text{ miles}
\]

\[
\theta \text{(in radians)} = \frac{S}{r}, \text{ so}
\]

\[
\theta \text{ (in radians)} = \frac{.0694 \text{ miles}}{.4 \text{ miles}} = 0.174 \text{ rads} = 0.174 \text{ rads} \cdot \frac{180^\circ}{\pi \text{ rads}} = 9.97^\circ \approx 10^\circ
\]

**Example:** What is the area of the sector of a circle of diameter 12.0 m subtended by an angle of 42 degrees?

**Solution:** A sector of a circle is “wedge” with a vertex at the circle’s center as shown below.
The fraction of a full circle that this sector represents is \( \frac{42^\circ}{360^\circ} = 0.1167 \). Thus, the area of the sector will be 11.67\% of the area of a full circle of radius 6.0 m, so
\[
A = 0.1167 \cdot \pi \cdot (6.0 \text{ m})^2 = 13.2 \text{ m}^2.
\]

**Example:** If a 34.0 in diameter tire on a car makes 400 rpm and never slips, how fast is the car moving?

**Solution:** If the tire never slips the distance the tire travels along the ground in one revolution is equal to the circumference of the tire. If the wheel makes 400 revolutions in one minute, then it is moving at a speed of 400 circumferences per minute.

\[
\text{Speed} = \frac{400 \text{ rev}}{1 \text{ min}} \cdot \frac{\pi \cdot 34.0 \text{ in}}{1 \text{ rev}} = \frac{42,700 \text{ in}}{\text{min}} = \frac{42,700 \text{ in}}{\text{min}} \cdot \frac{1 \text{ ft}}{12 \text{ in}} = \frac{3560 \text{ ft}}{\text{min}}
\]

\[
= \frac{3560 \text{ ft}}{\text{min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{1 \text{ mile}}{5280 \text{ ft}} = 40.6 \text{ mph}
\]
Your Turn!!

1. Convert the following angles from degree measure to radians and revolutions. Give answers to four decimal places.

<table>
<thead>
<tr>
<th>Angle (°)</th>
<th>Radians</th>
<th>Revolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>120°</td>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>300°</td>
<td>_______</td>
<td>_______</td>
</tr>
</tbody>
</table>

2. Convert the following angles from radian measure to degree measure and revolutions. Give answers to four decimal places.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Dec Degrees</th>
<th>DMS</th>
<th>Revolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>π/2</td>
<td>_______</td>
<td></td>
<td>_______</td>
</tr>
<tr>
<td>π/12</td>
<td>_______</td>
<td></td>
<td>_______</td>
</tr>
<tr>
<td>0.50</td>
<td>_______</td>
<td></td>
<td>_______</td>
</tr>
</tbody>
</table>

3. To three digits find the arclength subtended by an angle of 43.5° and a radius of 31.7 in.

4. Find the angle in both radians and degrees subtended by an arc of 40.0 cm in a circle of diameter 50.0 cm.

5. A car on a circular race track with a radius of 0.25 miles is travelling 110 mph. In six seconds through what central angle from the center of the track does the car turn?

6. Find the area of a sector of a circle of diameter 40.0 ft subtended by an angle of 16 degrees.

7. A bicyclist pedals at such a rate that both wheels rotate at 205 rpm. The outside wheel diameter is 26.0 in. Assuming that the tires never slip against the ground, what is the bicyclist’s speed in mph?
Section 5.3: The Volume and Surface Area of a Solid

The core idea of a volume is an area moved through a third dimension. Consider the following prism. This is a solid figure with two parallel faces and lateral (sideways) sides perpendicular to these faces. The parallel face is called the prism’s base. If the area of the base is $B$ and the perpendicular height between the two parallel faces is $h$, then the volume of the prism is given by the formula $V = h \cdot B$. Volume as the product of a length times an area must have units of cubic length, such as cubic inches, cubic feet etc. … .

![Prism Diagram]

The area occupied by the lateral sides of the prism is called the lateral surface area, $L$. The sum of the lateral surface area with the top and bottom base areas is called the total surface area, $A$. If you imagine “cutting” along the side of the prism perpendicular to the base and then “unfolding” and laying flat the lateral surface area, the resulting figure is a rectangle with dimensions equal to the base perimeter, $P$, and the height, $h$.

![Unfolded Lateral Surface Area]

**Example:** consider calculating the volume, lateral and total surface areas of the following prism.
The base is a triangle of sides 15.0 ft, 15.0 ft and 18.0 ft. The base area can be calculated using Heron’s formula.

\[
S = \frac{15.0 + 15.0 + 18.0}{2} \text{ ft} = 24.0 \text{ ft}
\]

\[
B = \sqrt{24.0 \times (24.0 - 15.0) \times (24.0 - 15.0) \times (24.0 - 18.0)} \text{ ft}^2
\]

\[
B = 108 \text{ ft}^2
\]

\[
V = 108 \text{ ft}^2 \times 9.0 \text{ ft} = 972 \text{ ft}^3
\]

\[
L = 48.0 \text{ ft} \times 9.0 \text{ ft} = 432 \text{ ft}^2
\]

\[
A = 432 \text{ ft}^2 + 2 \times 108 \text{ ft}^2 = 648 \text{ ft}^2
\]

An alternate approach to calculating this base area uses the fact that the base is an isosceles triangle. Dropping a perpendicular from the top vertex will then bisect the 18.0 ft side. The height of the resulting right triangle is then calculated by use of the Pythagorean Theorem.

\[
\sqrt{15^2 - 9^2} \text{ ft} = \sqrt{144} \text{ ft} = 12.0 \text{ ft}
\]

\[
B = \frac{1}{2} 18.0 \text{ ft} \times 12.0 \text{ ft} = 108 \text{ ft}^2
\]

The formulas for the volumes and surface areas of solids, which are not prisms, require more elaborate geometric reasoning. These are included in the results presented below.
Volume and Surface Area of a Rectangular Prism

Probably the most recognizable prism is the rectangular prism or “box”. The volume is simply the product of the lengths of the three sides.

![Diagram of a rectangular prism]

\[ B = \ell \cdot W \]
\[ P = 2\ell + 2W \]
\[ V = B \cdot h = \ell \cdot W \cdot h \]
\[ L = P \cdot h = 2\ell \cdot h + 2W \cdot h \]
\[ A = L + 2B = 2\ell \cdot h + 2W \cdot h + 2\ell \cdot W \]

Volume and Surface Area if a Cylinder

A right circular cylinder is also a prism. Here the circumference of the base circle is used in calculating the lateral surface area.

![Diagram of a cylinder]

\[ B = \pi r^2 = \frac{\pi D^2}{4} \]
\[ C = \pi D = 2\pi r \]
\[ V = B \cdot h = \pi r^2 h = \frac{\pi D^2 h}{4} \]
\[ L = C \cdot h = \pi Dh = 2\pi rh \]
\[ A = L + 2B = 2\pi rh + 2\pi r^2 = \pi Dh + \frac{\pi D^2}{2} \]
Volume of a Pyramid and Cone

Right pyramids and cones both have a volume equal to one third that of the corresponding prism having the same base and height.

\[ V = \frac{1}{3} h \cdot B = \frac{1}{3} h \cdot L \cdot W \quad V = \frac{1}{3} h \cdot B = \frac{1}{3} \pi r^2 h \]

Volume and Total Surface Area, A, of a Sphere

\[ V = \frac{4}{3} \pi r^3 = \frac{\pi D^3}{6} \]
\[ A = 4 \pi r^2 = \pi D^2 \]

Example: A storage tank is in the shape of a right circular cylinder capped with a hemispherical dome. The inner diameter of the tank is 15.0 ft and the height from the floor to the inner top of the hemisphere is 25.0 ft. The entire inside area of the tank needs to be coated with a sealant. One gallon of this sealant is required for every 150 square ft. What is the capacity of the tank in gallons? How many gallons of sealant are required to cover the inside surface of the tank?

Solution: The radius of the hemisphere must match the radius of the cylinder, \( r = 7.5 \) ft. The height of the cylinder is the total height, 25 ft, minus this radius, which amounts to 17.5 ft. The capacity or volume of the tank is the sum of the cylinder’s volume plus half of the volume of a sphere with a 7.5 ft radius. (To convert cubic ft to gallons see Unit 5 on Measurement.)

\[ V = \pi r^2 h + \frac{1}{2} \cdot \frac{4\pi r^3}{3} = \pi (7.5 \text{ ft})^2 \cdot 17.5 \text{ ft} + \frac{2\pi (7.5 \text{ ft})^3}{3} \]
\[ = 3976 \text{ ft}^3 = 3976 \text{ ft}^3 \cdot \frac{1 \text{ gal}}{0.13368 \text{ ft}^3} = 29,743 \text{ gal} \]
Given that the tank’s dimensions are only good to three digits, we should probably report the tank’s capacity as 29,700 gallons.

To determine how many gallons of sealant are required we need to calculate the total internal surface area of the tank. This consists of the circular floor, the lateral surface of the cylinder and half the surface area of a sphere with a 7.5 ft radius.

\[ A = \pi r^2 + 2\pi r \cdot h + \frac{1}{2} \cdot 4\pi r^2 = \pi (7.5 \text{ ft})^2 + 2\pi \cdot 7.5 \text{ ft} \cdot 17.5 \text{ ft} + 2\pi (7.5 \text{ ft})^2 = 1355 \text{ ft}^2 \]

One gallon of sealant covers 150 square feet of surface, so we will require \[ \frac{1355 \text{ ft}^2 \times \frac{1 \text{ gal}}{150 \text{ ft}^2}}{1} = 9.03 \text{ gal} \].

However, to be safe and to cover waste 10 gallons should probably be ordered.

**Example:** As a variation on the above problem, suppose we were designing a cylindrical storage tank (no hemispherical cap this time!) to have a capacity of 1 million gallons and a base diameter of 80 ft, how tall must the tank be?

![Diagram of a cylindrical tank](image)

**Solution:** First, we convert the million gallons into cubic feet.

\[ 1,000,000 \text{ gal} = 10^6 \text{ gal} \cdot \frac{0.13368 \text{ ft}^3}{1 \text{ gal}} = 133,680 \text{ ft}^3 = 134,000 \text{ ft}^3 \]

Then from the formula for the volume we solve for the height \( h \).

\[ V = B \cdot h = \pi r^2 \cdot h \]

\[ \frac{V}{\pi r^2} = \frac{\pi r^2 \cdot h}{\pi r^2} = h \quad \text{or} \quad h = \frac{V}{\pi r^2} = \frac{134000 \text{ ft}^3}{\pi (40 \text{ ft})^2} = 26.7 \text{ ft} \]
Your Turn!!

Find the lateral surface area, $L$, the total surface area, $A$, and the volume, $V$, of the following solids.

1. 

$L = \underline{\quad}$

$A = \underline{\quad}$

$V = \underline{\quad}$

2. 

$L = \underline{\quad}$

$A = \underline{\quad}$

$V = \underline{\quad}$
A cylindrical holding tank has an inner diameter of 40.0 ft and walls that are 18 in thick. The tank is designed to hold 350,000 gallons.

5. What is the tank’s inside perimeter?

6. What is the tank’s outside perimeter?

7. What is the area of the tank floor?

8. What is the height of the tank in feet?

9. A swimming pool is in the shape of a trapezoidal prism. The shallow end is 3.0 ft deep, the deep end is 8.0 ft, the width of the pool is 25 ft and the length from the shallow end to the deep end is 50 ft. How many gallons of water does it take to fill the pool?
Chapter 5 Practice Exam

1. Given the figure below and the fact that \( \angle A = 31^\circ \), find angles B and C.

   \[
   \begin{array}{c}
   \text{A} \\
   \text{B} \\
   \text{C}
   \end{array}
   \]

   \( B = \underline{\underline{\phantom{1}}} \)

   \( C = \underline{\underline{\phantom{1}}} \)

2. Find the perimeter, height, and area of the following triangle. State four digits in your answers for the height and area.

   \[
   \begin{array}{c}
   \text{A} \\
   \text{B} \\
   \text{C}
   \end{array}
   \]

   a. Perimeter \( \underline{\underline{\phantom{1}}} \)

   b. Height, \( h \) \( \underline{\underline{\phantom{1}}} \)

   c. Area \( \underline{\underline{\phantom{1}}} \)

3. Find the area and perimeter of the triangle below. Give the area to four digits.

   \[
   \begin{array}{c}
   \text{A} \\
   \text{B} \\
   \text{C}
   \end{array}
   \]

   a. Area \( \underline{\underline{\phantom{1}}} \) \ (Hint: use Heron’s Formula)

   b. Perimeter \( \underline{\underline{\phantom{1}}} \)
4. For a circle with a diameter of 12 inches, find the following

a. Area____________________ (Round answer to one decimal place)

b. Circumference______________ (Round answer to one decimal place)

c. The Arclength for of a sector of the circle mentioned above subtended by an angle of 10° (Remember that the angle must be in radians)

____________________ (Round answer to one decimal place)

d. Find the area of a sector formed by an angle of 10° for the circle above (Remember that the angle must be in radians)

____________________ (Round answer to one decimal place)

5. Convert 1.5300 radians to decimal degrees, DMS, and Revolutions

a. Decimal Degrees_________________ (Round answer to four decimal places)

b. DMS (Degrees, Minutes, Seconds)________________________

c. Revolutions____________________ (Round answer to four decimal places)

6. A right circular cylinder with base of radius 15 feet and a height of 3 feet, find:

a. Volume ____________ (Round answer to one decimal place)

b. Total Surface Area_________ (Round answer to one decimal place)
7. Find the volume of a cone with a base of diameter 5 feet and a height of 3 feet

Volume__________( Round answer to one decimal place )

8. Find the volume and total surface area of a cube with sides of length 11 feet

a. Volume____________ ( Round answer to one decimal place )

b. Total Surface Area___________ ( Round answer to one decimal place )

9. A sphere that you could just fit inside the cube in the previous problem, find:

a. Volume___________ ( Round answer to one decimal place )

b. Surface Area___________ ( Round answer to one decimal place )

c. Find the cost of covering the sphere in paint that cost $0.10 per square foot.
   ( Round answer to the nearest cent )
Chapter 6 - Trigonometry
Section 6.1 Sine, Cosine and Tangent

In this section we will restrict our attention to right triangles. With respect to the acute angles (the two angles less than 90 degrees) in a right triangle, we can classify the two legs (the sides which are not the largest side, the hypotenuse) as being either opposite to or adjacent to the given angle. This is shown below in the following two diagrams.

Since angles 1 and 2 are complementary, the value of one angle determines the second. Two triangles with the same set of angles are similar. They must have exactly the same shape. This shape then depends entirely on the value of one of the two acute angles. We say the shape is a “function” of the acute angle. The angle determines the shape and hence the various proportions between the triangle’s three sides. Thus these proportions are functions of the acute angle. We call these functions trigonometric functions (trig functions for short) and for a given acute angle there are three of special importance. These are the “sine”, “cosine” and “tangent” functions.
The two right triangles are similar (have exactly the same shape).

The common abbreviation for sine is “sin” (still pronounced sine). The cosine means the sine of the complement (see the picture above) and is abbreviated “cos”, while tangent is written as “tan”. Standard notation is to separate the name of the trig function from the name of the acute angle (called the “argument” of the function) with parenthesis. The idea is that, if you “input” the value of the acute angle to the trig function, it “outputs” the requested ratio of sides.

Definition of the Trigonometric Functions

\[
\sin(\text{acute } \angle) = \frac{\text{opposite side}}{\text{hypotenuse}}
\]

\[
\cos(\text{acute } \angle) = \frac{\text{adjacent side}}{\text{hypotenuse}}
\]

\[
\tan(\text{acute } \angle) = \frac{\text{opposite side}}{\text{adjacent side}}
\]
Consider the “famous” 3, 4, 5 right triangle. The values of the three trig functions are computed below for both of the acute angles A and B.

\[
\begin{align*}
\sin (\angle 2) &= \frac{b}{c} ; \quad \sin (\angle 1) = \frac{a}{c} = \cos (\angle 2) \\
\cos (\angle 2) &= \frac{a}{c} ; \quad \cos (\angle 1) = \frac{b}{c} = \sin (\angle 2) \\
\tan (\angle 2) &= \frac{b}{a} ; \quad \tan (\angle 1) = \frac{a}{b} = \frac{1}{\tan (\angle 2)}
\end{align*}
\]

Consider the “famous” 3, 4, 5 right triangle. The values of the three trig functions are computed below for both of the acute angles A and B.

\[
\begin{align*}
\sin (\angle A) &= \frac{4}{5} = 0.8 \quad ; \quad \sin (\angle B) = \frac{3}{5} = 0.6 \\
\cos (\angle A) &= \frac{3}{5} = 0.6 \quad ; \quad \cos (\angle B) = \frac{4}{5} = 0.8 \\
\tan (\angle A) &= \frac{4}{3} = 1.333... \quad ; \quad \tan (\angle B) = \frac{3}{4} = 0.75
\end{align*}
\]

To determine the values of these acute angles we need the “inverse” trig functions. A given acute angle uniquely determines the shape and thus the values of the ratios of the sides). So we can reverse this process and determine the acute angle from a given ratio of the sides. The angle is then a function of the ratio. These functions are most commonly designated as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. On most calculators they are accessed by using a 2nd, SHIFT, or INV key in combination with the primary trig function key. Using the inverse trig functions, angle A in the 3, 4, 5 triangle can be calculated by the various methods illustrated below. The results are shown to the nearest ten thousandth decimal degree and to the nearest second.
The following two examples using “special” triangles confirm that the inverse trig functions give the correct answer for the appropriate angle.

Consider an equilateral triangle of side 1. Dropping a perpendicular from the top vertex to the opposite side creates two right triangles with legs $\frac{1}{2}$ and $\sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$, and a hypotenuse of 1.

Now consider an isosceles right triangle with legs of length 1 and a hypotenuse of $\sqrt{1^2 + 1^2} = \sqrt{2}$. 
Your Turn!!

1. Convert the following angles from decimal degrees to DMS notation.

\[
\begin{array}{ll}
36.35^\circ & \text{DMS} \\
152.19^\circ & \\
262.25^\circ & \\
0.157^\circ & \\
\end{array}
\]

2. Convert the following angles from DMS notation to decimal degrees with four decimal places.

\[
\begin{array}{ll}
15^\circ42'19" & \text{Decimal Degree} \\
121^\circ37'53" & \\
1^\circ16'29" & \\
3'14" & \\
\end{array}
\]

For each angle give the values of the three trig functions to four decimal places.

\[
\begin{array}{ll}
\cos \theta = & 49.76^\circ \\
\sin \theta = & \\
\tan \theta = & \\
\cos \theta = & 16^\circ11'49" \\
\sin \theta = & \\
\tan \theta = & \\
\end{array}
\]
5. Given that $\sin \theta = 0.55$ and $0^\circ < \theta < 90^\circ$, determine the following:

<table>
<thead>
<tr>
<th>DD (4 places)</th>
<th>DMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = $</td>
<td></td>
</tr>
<tr>
<td>$\cos \theta = $</td>
<td></td>
</tr>
<tr>
<td>$\sin \theta = $</td>
<td></td>
</tr>
<tr>
<td>$\tan \theta = $</td>
<td></td>
</tr>
</tbody>
</table>

6. Given that $\tan \theta = 1.07$ and $0^\circ < \theta < 90^\circ$, determine the following:

<table>
<thead>
<tr>
<th>DD (4 places)</th>
<th>DMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = $</td>
<td></td>
</tr>
<tr>
<td>$\cos \theta = $</td>
<td></td>
</tr>
<tr>
<td>$\sin \theta = $</td>
<td></td>
</tr>
<tr>
<td>$\tan \theta = $</td>
<td></td>
</tr>
</tbody>
</table>

**Section 6.2 Solving Right Triangles**

With the three trig functions and their inverses we can solve for the missing information (lengths of sides and measure of angles) in any right triangle provided that we know either two sides or one side and one of the acute angles. To facilitate such problems the following notation will be used to describe any triangle. The **CAPITAL** letters $A$, $B$, and $C$ will designate the three angles. The corresponding lower case letter will represent the side opposite to each angle. This is illustrated in the figure below.
For a right triangle $\angle C = 90^\circ$ and $c$ is the hypotenuse.

In any triangle the largest side is opposite the largest angle and the smallest side is opposite the smallest angle.

The following three examples demonstrate the solution of right triangle problems.

Example 1

\[
a = \_\_\_\_\_\_\_\_\_
\]

\[
b = \_\_\_\_\_\_\_\_\_
\]

\[
c = 7.56 \text{ m}
\]

\[
A = 36.5^\circ
\]

\[
B = \_\_\_\_\_\_\_\_\_
\]

\[
C = 90^\circ
\]

**Solution:** Since angles $A$ and $B$ are complementary, $B = 90^\circ - 36.5^\circ = 53.5^\circ = 53^\circ 30'$.

From the definition of the sine,

\[
\sin(36.5^\circ) = \frac{\text{opp}}{\text{hyp}} = \frac{a}{7.56 \text{ m}} \Rightarrow \text{solving for } a, \quad \frac{a \cdot 7.56 \text{ m}}{7.56 \text{ m}} = 7.56 \text{ m} \cdot \sin(36.5^\circ) = 4.497 \text{ m}
\]

so $a = 4.497 \text{ m}$.

The length of side $b$ can now be determined using the cosine function.

\[
\cos(36.5^\circ) = \frac{\text{adj}}{\text{hyp}} = \frac{b}{7.56 \text{ m}} \Rightarrow \text{solving for } b, \quad \frac{b \cdot 7.56 \text{ m}}{7.56 \text{ m}} = 7.56 \text{ m} \cdot \cos(36.5^\circ) = 6.077 \text{ m}
\]

so $b = 6.077 \text{ m}$.

There are two other methods of solution for $b$ which use the calculated value of $a$ and either the Pythagorean Theorem or the tangent function. However, any method which uses calculated values rather than the initial “given” values (the values of $c$, $C$, and $A$) risks unintended inaccuracy due to rounding errors. Since the solution shown above uses **only** the given information, it is the preferred method of solution.
Example 2

\[ a = 12.0 \text{ ft} \quad b = \quad c = \quad \]

\[ A = \quad B = 19^\circ 23'39'' \quad C = 90^\circ \]

**Solution:** Angle \( A \) is the complement of angle \( B \), so
\[ A = 90^\circ - B = 90^\circ - 19^\circ 23'39'' = 89^\circ 59'60'' - 19^\circ 23'39'' = 70^\circ 36'21''. \]

Next we use the tangent function to calculate the length of side \( b \), then the cosine to calculate the hypotenuse, \( c \).

\[ 19^\circ 23'39'' = 19.39417^\circ \]
\[ \tan(19.39417^\circ) = \frac{b}{12.0 \text{ ft}} \Rightarrow b = 12.0 \text{ ft} \cdot \tan(19.39417^\circ) = 4.224 \text{ ft}. \]

Again other solution methods can be used, but the approach given above solves for all unknowns solely in terms of the initial given information.

Example 3

\[ a = 23.3 \text{ cm} \quad b = \quad c = 37.8 \text{ cm} \]

\[ A = \quad B = \quad C = 90^\circ \]

**Solution:** First use the Pythagorean Theorem to calculate the length of side \( b \), then use the inverse sine to calculate one of the missing acute angles.

\[ b = \sqrt{c^2 - a^2} = \sqrt{37.8^2 - 23.3^2} \text{ cm} = 29.7649 \text{ cm}. \]
\[ \sin(A) = \frac{a}{c} = \frac{23.3}{37.8} = 0.61640 \Rightarrow A = \sin^{-1}(0.61640) = 38.0539^\circ = 38^\circ 03'14''. \]
\[ B = 90^\circ - A = 90^\circ - 38.0539^\circ = 51.9461^\circ = 51^\circ 56'46''. \]

In the above examples more digits were reported in the computed answers than the precision of the given data warranted. This was done to illustrate the use of the calculator in solving these problems and to avoid round off discrepancies when comparing different approaches. In practice, the computed answers would be reported to a fewer number of digits.

In law enforcement the solution of right triangles is used to calculate the actual speed of a vehicle from the radar reading of its speed. This is often referred to as the “cosine angle error”.

Consider the diagram shown below. The angle between the direction of the radar gun and the road along which the vehicle is moving is given as theta. The radar gun actually measures the “component” of the vehicle’s speed along its direction of sight. So unless theta is zero (i.e., unless the gun is aimed parallel to the road), the speed determined by radar is actually less than the actual speed of the vehicle.
Let $s$ stand for the speed measured with the radar gun and let $v$ be the actual speed of the vehicle along the road. Then from trigonometry we have the relations stated below.

$$s = v \cdot \cos(\theta) \quad \text{or} \quad v = \frac{s}{\cos(\theta)}$$

For example, if a the radar gun makes a 20 degree angle with the road and reads 85 mph, the actual speed of the car road is calculated as follows:

$$v = \frac{85 \text{ mph}}{\cos(20^\circ)} = 90.5 \text{ mph}$$

Similarly, if the car is travelling at 65 mph and the radar gun makes a 45 degree angle with the road, then the speed as read from the radar unit is given by

$$s = 65 \text{ mph} \cdot \cos(45^\circ) = 46.0 \text{ mph}.$$

Your Turn!!

Solve for the missing sides (3 digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

1. $a = 16.0 \text{ cm} \quad b = 12.0 \text{ cm} \quad c = \underline{\phantom{000}}$

   $A = \underline{\phantom{000}} \quad B = \underline{\phantom{000}} \quad C = 90.00^\circ$

2. $a = \underline{\phantom{000}} \quad b = \underline{\phantom{000}} \quad c = 5.0 \text{ cm}$

   $A = 40.00^\circ \quad B = \underline{\phantom{000}} \quad C = 90.00^\circ$

3. $a = 25.4 \text{ ft} \quad b = \underline{\phantom{000}} \quad c = \underline{\phantom{000}}$

   $A = 17.21^\circ \quad B = \underline{\phantom{000}} \quad C = 90.00^\circ$
4. A radar gun makes a 32 degree angle with a road and measures a speed of 59.4 mph. How fast is the car actually traveling?

5. If a radar gun is positioned at a 35 degree angle with respect to the road and the speed limit is 35 mph, what minimum radar reading means an actual speed more than 10 mph over the speed limit?

6. If a car’s actual speed is 60 mph, fill in the following table. Give answers to the nearest tenth of a mile per hour.

<table>
<thead>
<tr>
<th>Angle between direction of radar gun and road</th>
<th>Radar reading of car’s speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td></td>
</tr>
<tr>
<td>10°</td>
<td></td>
</tr>
<tr>
<td>20°</td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td></td>
</tr>
<tr>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>50°</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td></td>
</tr>
<tr>
<td>70°</td>
<td></td>
</tr>
<tr>
<td>80°</td>
<td></td>
</tr>
<tr>
<td>90°</td>
<td></td>
</tr>
</tbody>
</table>
Section 6.3 The Law of Sines and the Law of Cosines

In order to apply the trigonometric functions to a wider variety of problems we need ways of solving oblique (i.e., non-right) triangles. The two tools that allow us to proceed are the Law of Sines and the Law of Cosines. However, we first need to extend the definition of the trig functions from acute angles to any angle. To do this we construct a circle of radius 1 centered at the origin of an \( x - y \) set of coordinates. As shown below this circle crosses the axes at the points \((1, 0), (0, 1), (-1, 0), (0, -1)\).

From the origin construct a line segment that intersects the circle at a point with coordinates \((x, y)\). The angle theta is measured from the positive \(x\) axis to this segment. The trig functions are then defined as follows:

\[
\cos(\theta) = \frac{x}{1} = x
\]

\[
\sin(\theta) = \frac{y}{1} = y
\]

\[
\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0
\]

If both \(x\) and \(y\) are positive theta is an acute angle and the definitions given above are just the results for a right triangle with adjacent leg \(x\), opposite leg \(y\), and a hypotenuse of 1. If either \(x\) or \(y\) is negative or zero, then theta is not an acute angle and we obtain an extension of our earlier definitions. For example, we now have the following results.

\[
\cos(270^\circ) = 0 \quad \cos(180^\circ) = -1 \quad \cos(0^\circ) = 1 \quad \cos(90^\circ) = 0
\]

\[
\sin(270^\circ) = -1 \quad \sin(180^\circ) = 0 \quad \sin(0^\circ) = 0 \quad \sin(90^\circ) = 1
\]

\[
\tan(270^\circ) \text{ is undefined} \quad \tan(180^\circ) = 0 \quad \tan(0^\circ) = 0 \quad \tan(90^\circ) \text{ is undefined}
\]

If an angle is negative, it is measured “down” from the \(x\) axis, i.e., the angle moves “clockwise” around the circle. For negative inputs, the inverse sine and inverse tangent functions will return an angle between negative ninety degrees and zero degrees, meaning that the angle is in that part of the coordinate system where \(x\) is positive and \(y\) is negative. On the other hand, for a negative input, the inverse cosine function will return an angle between positive ninety degrees and positive one hundred eighty degrees (that is, an obtuse angle). This means that such an angle is in that part of the coordinate system where \(x\) is negative and \(y\) is positive. **Note:**
Even though the inverse cosine function can return an obtuse angle, the inverse sine cannot do so. Thus, we get the following results.

\[
\begin{align*}
\cos^{-1}(-0.707106781) &= 135^\circ \\
\sin^{-1}(-0.707106781) &= -45^\circ \\
\tan^{-1}(-1) &= -45^\circ
\end{align*}
\]

As the diagram below illustrates if \( \cos(\theta) = -x \) and \( \sin(\theta) = y \), then
\[
\cos(180^\circ - \theta) = x \quad \text{and} \quad \sin(180^\circ - \theta) = y
\]

or
\[
\begin{align*}
\cos(180^\circ - \theta) &= -\cos(\theta) \\
\sin(180^\circ - \theta) &= \sin(\theta)
\end{align*}
\]

Finally, by the Pythagorean Theorem, for any point with coordinates \((x, y)\) on this circle, we have the equation.

\[
x^2 + y^2 = 1
\]

or for any angle \(\theta\)
\[
[\cos(\theta)]^2 + [\sin(\theta)]^2 = 1
\]

The use of trig functions defined for all angles is useful in navigation. The standard notation is to take North as the direction of the positive \(y\) axis and East as the direction of the positive \(x\) axis. If the angle \(\theta\) is measured from East, then the positions east and north of the origin (the starting position) are given respectively by

\[
x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta)
\]

Where \(r\) is the distance from the origin. If \(x\) is negative, the position is to the west and if \(y\) is negative the position is to the south. A special notation is also used to specify angles. For example, in the diagram below the orientation could be specified as a direction-angle-direction such as \(E53.7^\circ N\) or \(N36.3^\circ E\). This is also sometimes stated as 53.7 degrees north of east or 36.3 degrees east of north.
In the diagram below the orientation could be specified in any of the following equivalent ways.

For example, consider the following problem. If a plane flies W18.6°N at a speed of 347 mph for 33 minutes, how far west and north has the plane traveled?

**Solution:**

\[
\theta = 180^\circ - 18.6^\circ = 161.4^\circ \\
\theta = 180^\circ - 18.6^\circ - 180^\circ = 32.8^\circ \\
\theta = 180^\circ - 47.2^\circ = 132.8^\circ \\
\theta = 180^\circ - 90^\circ = 90^\circ \\
\theta = 212.8^\circ \\
\theta = -147.2^\circ \\
W(212.8^\circ - 180^\circ)S = W32.8^\circ S \\
S(47.2^\circ - 90^\circ)W = S57.2^\circ W
\]
So the plane has traveled a total distance of 191 miles. It is 181 miles west and 60.9 miles north of its original position.

The Law of Sines can be explained by the following argument. Consider an acute (i.e., the largest angle is less than a right angle) triangle. Drop a perpendicular of length $h$ from the top vertex to the bottom base.

Since the labeling of the sides and opposite angles is completely arbitrary this last ratio must also equal the ratio of the sine of angle $B$ to the length of side $b$. If the triangle were obtuse (i.e., the largest angle is larger than a right angle) the same result is obtained as shown below.

Thus, for any triangle we have that the ratio of the sine of any internal angle to the length of the opposite side is a constant for that specific triangle. The Law of Sines is summarized below.

\[
\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \quad \text{or} \quad \frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}
\]

To justify the Law of Cosines again consider the acute triangle shown. Drop a perpendicular of length $h$ to the bottom base. Using the Pythagorean Theorem on both of the right triangles formed yields the following equations.
Again since the labeling of sides and angles was arbitrary, similar results for the square of sides $a$ and $b$ must also be true. If the triangle is obtuse, we still obtain the same results as shown below.

\[ \begin{align*}
    c^2 &= (b-x)^2 + h^2 \\
    (b-x)^2 &= (b-x) \cdot (b-x) \\
    &= (b-x) \cdot b + (b-x) \cdot (-x) \\
    &= b \cdot (b-x) - x \cdot (b-x) \\
    &= b \cdot b + x \cdot (-x) + b \cdot (-x) - x \cdot (-x) \\
    &= b^2 - b \cdot x - b \cdot x + x^2 = b^2 - 2b \cdot x + x^2 \\
    \text{But, } x &= a \cdot \cos(C) \text{ and } x^2 + h^2 = a^2 \\
    \text{So } c^2 &= b^2 - 2b \cdot x + x^2 + h^2 = b^2 - 2b \cdot a \cdot \cos(C) + a^2 \\
    c^2 &= a^2 + b^2 - 2a \cdot b \cdot \cos(C) \\
\end{align*} \]

In summary, the Law of Cosines states that for any triangle the length of any side squared equals the sum of the squares of the other two sides minus twice the product of the other two sides with the cosine of the opposite angle. The Law of Cosines generalizes the Pythagorean Theorem to any kind of triangle, not just those with a right angle.

\[ \begin{align*}
    c^2 &= a^2 + b^2 - 2a \cdot b \cdot \cos(C) \Rightarrow c = \sqrt{a^2 + b^2 - 2a \cdot b \cdot \cos(C)} \\
    b^2 &= a^2 + c^2 - 2a \cdot c \cdot \cos(B) \Rightarrow b = \sqrt{a^2 + c^2 - 2a \cdot c \cdot \cos(B)} \\
    a^2 &= b^2 + c^2 - 2b \cdot c \cdot \cos(A) \Rightarrow a = \sqrt{b^2 + c^2 - 2b \cdot c \cdot \cos(A)} \\
\end{align*} \]

Rearranging these formulas to solve for the relevant angles gives the following results.
\[
c^2 = a^2 + b^2 - 2ab \cdot \cos(C) \Rightarrow c^2 - a^2 - b^2 = -2ab \cdot \cos(C)
\]
\[
\frac{c^2 - a^2 - b^2}{2a} = -2ab
\]
\[
\cos(C) = -\frac{c^2 - a^2 - b^2}{2ab} = (-1) \cdot \frac{(c^2 - a^2 - b^2)}{2a} = \frac{a^2 + b^2 - c^2}{2a}
\]
\[
\cos(C) = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow C = \cos^{-1}\left(\frac{(a^2 + b^2 - c^2)}{2ab}\right)
\]
\[
\cos(B) = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow B = \cos^{-1}\left(\frac{(a^2 + c^2 - b^2)}{2ac}\right)
\]
\[
\cos(A) = \frac{b^2 + c^2 - a^2}{2bc} \Rightarrow A = \cos^{-1}\left(\frac{(b^2 + c^2 - a^2)}{2bc}\right)
\]

The unneeded parentheses were added to the formulas for the sides \(a, b, c\) and the angles \(A, B\) and \(C\) to emphasize that both the square root symbol and the fraction bar act as implied grouping symbols. Failure to provide these parentheses on a scientific calculator will result in a wrong answer.

Since the inverse cosine function returns angles in the range \(0^\circ\) to \(180^\circ\), the Law of Cosines can be used to find any angle of a triangle when the lengths of all three sides are known. In contrast, since the inverse sine function can’t return angles between \(90^\circ\) to \(180^\circ\), the Law of Sines will be unable to directly compute the measure of an obtuse angle in a triangle.

**Your Turn!!**

A plane travels \(W35^\circ N\) (i.e., 35 degrees north of west) for 15 minutes at 350 mph.

1. In these 15 minutes how far did the plane travel?
2. In these 15 minutes how far west did the plane travel?
3. In these 15 minutes how far north did the plane travel?

4. Given that \(\sin \theta = 0.35\) and \(90^\circ < \theta < 180^\circ\), determine the following:
   \[
   \theta = ____________ \\
   \cos \theta = ____________ \\
   \sin \theta = ____________ \\
   \tan \theta = ____________
   \]

5. Given that \(\cos \theta = -0.55\) and \(90^\circ < \theta < 180^\circ\), determine the following:
6. Given that $\tan \theta = 1.25$ and $180^\circ < \theta < 270^\circ$, determine the following:

<table>
<thead>
<tr>
<th>DD (4 places)</th>
<th>DMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td></td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td></td>
</tr>
<tr>
<td>$\sin \theta$</td>
<td></td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td></td>
</tr>
</tbody>
</table>

Section 6.4 Solving Oblique Triangles

With the sine and cosine functions and their inverses we can solve for the three missing items of information (lengths of sides or measure of angles) in any triangle provided that we have the three relevant inputs. The three relevant inputs are based on the congruency criteria discussed in Chapter 6. Specifically, these are SAS, ASA, and SSS. The procedure for solving a triangle in each case is presented below.

1. SAS

Given two sides $b$ and $c$ and the angle $A$ between these sides:

A. Use the Law of Cosines to calculate the length of the side opposite the given angle.

$$a^2 = b^2 + c^2 - 2b \cdot c \cdot \cos(A) \Rightarrow a = \sqrt{b^2 + c^2 - 2b \cdot c \cdot \cos(A)}$$
B. Use the Law of Sines to calculate the measure of the angle opposite to the smaller of sides $a$ and $b$. This is done to avoid trying to compute an obtuse angle using the inverse sine function. Then calculate the third missing angle by subtracting the sum of the two known angles from $180^\circ$.

If $b > c \quad \frac{\sin(C)}{c} = \frac{\sin(A)}{a} \quad \Rightarrow \quad \sin(C) = \frac{c \cdot \sin(A)}{a} \quad \Rightarrow \quad C = \sin^{-1}\left(\frac{c \cdot \sin(A)}{a}\right) \quad B = 180^\circ - A - C$

If $c > b \quad \frac{\sin(B)}{b} = \frac{\sin(A)}{a} \quad \Rightarrow \quad \sin(B) = \frac{b \cdot \sin(A)}{a} \quad \Rightarrow \quad B = \sin^{-1}\left(\frac{b \cdot \sin(A)}{a}\right) \quad C = 180^\circ - A - B$

**Note:** In the formulas above we used the version of the Law of Sines with the sines of the unknown angles in the numerator. This is not at all necessary, but it does make the algebra of solving for the unknown angles a little easier to follow.

---

2. **ASA**

Given two angles $A$ and $B$ and $c$ the side between them:

A. Calculate the missing angle by subtracting the sum of the two given angles from 180 degrees.

B. Use the Law of Sines to calculate the two missing sides.

\[ C = 180^\circ - A - B \]

\[ \frac{a}{\sin(A)} = \frac{c}{\sin(C)} \quad \Rightarrow \quad a = \frac{c \cdot \sin(A)}{\sin(C)} \]

\[ \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \quad \Rightarrow \quad b = \frac{c \cdot \sin(B)}{\sin(C)} \]
Note: In the formulas above we used the version of the Law of Sines with the unknown sides in the numerator.

3. SSS

Given the three sides, \( a, b, c \), determine the three angles \( A, B, C \):

A. Use the Law of Cosines to calculate the measure of the angle opposite to the longest side. This is the only angle in the triangle that could be obtuse. Hence this angle can always be calculated using the inverse cosine function, but may not be directly computable using the inverse sine function.

\[
C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)
\]

B. Use the Law of Sines to calculate the measure of one of the remaining two unknown angles. Then calculate the third missing angle by subtracting the sum of the two known angles from 180 degrees.

\[
\frac{\sin(A)}{a} = \frac{\sin(C)}{c} \Rightarrow \sin(A) = \frac{a \cdot \sin(C)}{c} \Rightarrow A = \sin^{-1}\left(\frac{a \cdot \sin(C)}{c}\right)
\]

\[
B = 180^\circ - A - C
\]

Example 1

Given a triangle with the following information, solve for the missing side and angles.

\[
a = 3.85 \text{ ft} \quad b = 4.25 \text{ ft} \quad c = \quad \text{__________}
\]

\[
A = \quad \text{__________} \quad B = \quad \text{__________} \quad C = 67^\circ
\]

Solution: This is a SAS problem. First we calculate the missing side \( c \), then the angles \( A \) and \( B \).
Example 2

Given a triangle with the following information, solve for the missing sides and angle.

\[ a = \underline{\hspace{2cm}} \quad b = \underline{\hspace{2cm}} \quad c = 88.9 \text{ mm} \]

\[ A = 43^\circ \quad B = 59^\circ \quad C = \underline{\hspace{2cm}} \]

Solution: This is an ASA problem. First we calculate the missing angle \( C \), then the sides \( a \) and \( b \).

\[ C = 180^\circ - A - B = 180^\circ - 43^\circ - 59^\circ = 78^\circ \]

\[ \frac{a}{\sin(A)} = \frac{c}{\sin(C)} \Rightarrow a = \frac{c \sin(A)}{\sin(C)} = \frac{88.9 \text{ mm} \times \sin(43^\circ)}{\sin(78^\circ)} = 61.98 \text{ mm}. \]

\[ \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \Rightarrow b = \frac{c \sin(B)}{\sin(C)} = \frac{88.9 \text{ mm} \times \sin(59^\circ)}{\sin(78^\circ)} = 77.90 \text{ mm}. \]

Example 3

Given a triangle with the following information, solve for the three missing angles.

\[ a = 9.56 \text{ in} \quad b = 3.67 \text{ in} \quad c = 8.19 \text{ in} \]

\[ A = \underline{\hspace{2cm}} \quad B = \underline{\hspace{2cm}} \quad C = \underline{\hspace{2cm}} \]
Solution: This is an SSS problem. First we calculate the missing angle $A$ (the largest angle), then the angles $C$ and $B$.

\[
A = \cos^{-1}\left( \frac{b^2 + c^2 - a^2}{2 \cdot b \cdot c} \right) = \cos^{-1}\left( \frac{3.67^2 + 8.19^2 - 9.56^2}{2 \times 3.67 \times 8.19} \right) = \cos^{-1}(-0.18046) = 100.40^\circ
\]

\[
\frac{\sin(C)}{c} = \frac{\sin(A)}{a} \Rightarrow \frac{\sin(C)}{c} = \frac{8.19 \text{ in} \times \sin(100.40^\circ)}{9.56 \text{ in}} = 0.84263
\]

\[
C = \sin^{-1}(0.84263) = 57.42^\circ
\]

\[
B = 180^\circ - A - C = 180^\circ - 100.40^\circ - 57.42^\circ = 22.18^\circ
\]

Note: If first we had calculated either angle $B$ or $C$ from the Law of Cosines and then had attempted to calculate angle $A$ using the Law of Sines, we would have instead obtained $A$’s supplement. The reason for this is again the fact that for positive input, the inverse sine function always returns an angle between zero and ninety degrees. This illustrates the need to set up the calculation in such a way so that the Law of Sines is not used to calculate the angle opposite to the longest side. Such an incorrect solution is shown below.

\[
B = \cos^{-1}\left( \frac{a^2 + c^2 - b^2}{2 \cdot a \cdot c} \right) = \cos^{-1}\left( \frac{9.56^2 + 8.19^2 - 3.67^2}{2 \times 9.56 \times 8.19} \right) = \cos^{-1}(0.92597) = 22.18^\circ
\]

\[
\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \Rightarrow \frac{\sin(A)}{a} = \frac{9.56 \text{ in} \times \sin(22.18^\circ)}{3.67 \text{ in}} = 0.98358
\]

So, "$A" = \sin^{-1}(0.98358) = 79.60^\circ = 180^\circ - 100.40^\circ"

Using this flawed procedure we obtained the supplement of the correct answer. Since \( \sin(79.60^\circ) = \sin(100.40^\circ) = 0.98358 \), the inverse sine function gave the “wrong" answer.

Example 4
An explorer is taking measurements on a distant mountain peak. She initially measures an angle of elevation of $37^\circ 44' 06"$. She then advances 500 meters closer and measures a new angle of elevation of $39^\circ 22' 06"$. How high is the mountain peak above the horizontal of the explorer?
Solution: Angles of elevation are measured from the horizontal “up” to a distant object. In a similar way angles of depression are measured from the horizontal “down” to a distant object.

To determine the height \( h \), we need to determine either one of the lengths \( a \) or \( b \) and use right triangle trigonometry. The determination of \( a \) or \( b \) means solving an ASA triangle. First we calculate the angle \( C \), then use the Law of Sines to calculate either \( a \) or \( b \).

\[
\begin{align*}
A &= 37^\circ 44' 06'' = 37.735^\circ \\
B &= 180^\circ - 39^\circ 22' 06'' = 140^\circ 37' 54'' = 140.63167^\circ \\
C &= 180^\circ - (A + B) = 180^\circ - 178^\circ 22' 1'' 38'' = 1.6333^\circ \\
\end{align*}
\]

\[
\frac{a}{\sin(A)} = \frac{c}{\sin(C)} \Rightarrow a = \frac{c \sin(A)}{\sin(C)} = \frac{500 \times \sin(37.735^\circ)}{\sin(1.6333^\circ)} = 10,736 \text{ m}
\]

So, \( h = a \cdot \sin(39.3683^\circ) = 10,736 \text{ m} \times \sin(39.3683^\circ) = 6810 \text{ m} \)

or

\[
\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \Rightarrow b = \frac{c \sin(B)}{\sin(C)} = \frac{500 \times \sin(140.6317^\circ)}{\sin(1.6333^\circ)} = 11,127 \text{ m}
\]

So, \( h = b \cdot \sin(37.735^\circ) = 11,127 \text{ m} \times \sin(37.735^\circ) = 6810 \text{ m} \)

Your Turn!!

Solve for the missing sides (3 digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

1. \( a = 17.2 \text{ cm} \) \( b = 21.3 \text{ cm} \) \( c = \) ____________
   \[A = \] ____________ \[B = \] ____________ \[C = 45.00^\circ\]

2. \( a = 11.1 \text{ in} \) \( b = 21.5 \text{ in} \) \( c = 16.2 \text{ in} \)
   \[A = \] ____________ \[B = \] ____________ \[C = \] ____________

3. \( a = \) ____________ \( b = \) ____________ \( c = 14.3 \text{ ft} \)
   \[A = 50.00^\circ \] \[B = 30.00^\circ \] \[C = \] ____________

4. A geologist sights a distant hilltop with an angle of elevation of 23 degrees 37 minutes. She then advances 1000 m closer to the mountain and measures a new angle of elevation of 29 degrees 10 minutes. How high is the hilltop above the level of the geologist?

5. Two airplanes leave the same airport at the same time. The planes are travelling 400 mph and 360 mph respectively. After one and one half-hours the planes are 200 miles apart. To two decimal places what is the angle between the airplanes’ courses of flight?
Chapter 6 Practice Exam

1. Convert the following angles from decimal degrees to DMS notation and from DMS to decimal degrees to the nearest ten-thousandth.

<table>
<thead>
<tr>
<th>Decimal Degree</th>
<th>DMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>11°30’</td>
<td>__________</td>
</tr>
<tr>
<td>83°19’10”</td>
<td>__________</td>
</tr>
<tr>
<td>50°50”</td>
<td>__________</td>
</tr>
<tr>
<td>42°15’30”</td>
<td>__________</td>
</tr>
</tbody>
</table>

2. For each angle give the values of the three trig functions to four decimal places.

\[ \theta = 52°16’49” \]
\[
\begin{align*}
\cos \theta &= \quad \quad \\
\sin \theta &= \quad \quad \\
\tan \theta &= \quad \quad \\
\end{align*}
\]

\[ \theta = 20.25° \]
\[
\begin{align*}
\cos \theta &= \quad \quad \\
\sin \theta &= \quad \quad \\
\tan \theta &= \quad \quad \\
\end{align*}
\]

3. Given that \( \sin \theta = 0.25 \) and \( 0° < \theta < 90° \), determine the following:

\[
\begin{align*}
\theta &= \quad \quad \\
cos \theta &= \quad \quad \\
\sin \theta &= \quad \quad \\
\tan \theta &= \quad \quad \\
\end{align*}
\]

4. Given that \( \tan \theta = 0.85 \) and \( 0° < \theta < 90° \), determine the following:

\[
\begin{align*}
\theta &= \quad \quad \\
cos \theta &= \quad \quad \\
\sin \theta &= \quad \quad \\
\tan \theta &= \quad \quad \\
\end{align*}
\]

5. Given that \( \cos \theta = -.85 \) and \( 90° < \theta < 180° \), determine the following:

\[
\begin{align*}
\theta &= \quad \quad \\
cos \theta &= \quad \quad \\
\sin \theta &= \quad \quad \\
\tan \theta &= \quad \quad \\
\end{align*}
\]
6. Given that \( \tan \theta = 0.75 \) and \( 180^\circ < \theta < 270^\circ \), determine the following:

\[
\begin{array}{c|c}
\theta & \text{DMS} \\
\hline
\theta & \\
\cos \theta & \\
\sin \theta & \\
\tan \theta & \\
\end{array}
\]

Solve for the missing sides (3 digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

7. \( a = 23.0 \text{ cm} \) \( b = 18.5 \text{ cm} \) \( c = \) ______________________

\( A = \) _______________ \( B = \) _______________ \( C = 90.00^\circ \)

8. \( a = \) _______________ \( b = \) _______________ \( c = 86.7 \text{ ft} \)

\( A = 23.75^\circ \) \( B = \) _______________ \( C = 90.00^\circ \)

9. \( a = 3.68 \text{ m} \) \( b = \) _______________ \( c = \) ______________________

\( A = 55.86^\circ \) \( B = \) _______________ \( C = 90.00^\circ \)

10. \( a = 25.0 \text{ cm} \) \( b = 33.6 \text{ cm} \) \( c = \) ______________________

\( A = \) _______________ \( B = \) _______________ \( C = 120.00^\circ \)

11. \( a = 42.0 \text{ in} \) \( b = \) _______________ \( c = \) ______________________

\( A = 38.41^\circ \) \( B = \) _______________ \( C = 91.50^\circ \)

12. \( a = 19.6 \text{ cm} \) \( b = 28.6 \text{ cm} \) \( c = 43.2 \text{ cm} \)

\( A = \) _______________ \( B = \) _______________ \( C = \) _______________

13. A radar gun makes a 45 degree angle with a road and measures a speed of 64.8 mph. How fast is the car actually traveling?

14. Two airplanes leave the same airport at the same time. The planes are traveling 500 mph and 480 mph respectively. After one half-hour the planes are 160 miles apart. To two decimal places what is the angle between the planes’ lines of flight.
Chapter 7 - Statistics

Statistics is the science of collecting, organizing, analyzing, presenting, and interpreting data.

The data that is used in statistics can come to us from many sources. We can either do the data collection ourselves or the data can be obtained from some other source like a newspaper or one of the various levels of government or from a business that collects information about its customers.

There are many different ways in which we can talk about data. We will look at two major classifications of data: qualitative versus quantitative and discrete versus continuous.

Data can be considered to be either qualitative or quantitative. Qualitative data is data that is a name or reference to something to someplace or is a characteristic of something. Examples of qualitative data are a person’s hair color or eye color, type of car driven, the letter grade received in a class or the objects shape. Qualitative data can also be numeric; that is, it can contain numbers. For example a zip code is a qualitative data value because it is a reference to a geographical location in the United States. A ranking of 1 through 10 is also qualitative in nature because it is a reference to how well something is being done. Quantitative data is data that is numerical and for which it makes sense to add, subtract, multiply or divide data values. Some examples of quantitative data are the speeds at which cars pass a certain point on a road, the height of a male human, and the number of 8-ounce cups of coffee a person buys in a day. Zip codes and phone numbers are numeric in nature but it would not make any sense to do arithmetic on these numbers and that is why they are qualitative.

Quantitative data can be further defined as being either discrete or continuous. Discrete data is data that can be counted. Some examples of discrete data are how many commercials there are in one hour of television programming, how many cars pass a given point on a road in a day, and how many letters are mailed to a certain zip-code in a week. Continuous data is data that can be measured. Some examples of continuous data are the length of a television commercial, how fast the car are travelling past a given point on a road, and the weight of the letters that are mailed to a certain zip code in a week.

Data collection is the hard part of doing a statistical analysis. Before we begin we must have a data collection plan. That is, we must consider what kind of information we are trying to gather and how are we going to collect this information. We can collect data by direct observation or by means of a survey or we can get our data from another source. Direct observation means that we do the actual measurement or count of the subjects. If we are interested in the temperature at which certain liquids boil then we must be there to insert the thermometer into the liquid and read the temperature at which the liquid boils. If we are going to use a survey then we must design the survey in such a way that the questions we ask are not misleading in nature, have a limited number of possible responses, and are not overly burdensome to complete. If we get our data from some other source we must ensure that the source is trustworthy and that the data is what we wanted and fits within the framework of the analysis we are doing.

Section 7.1 Organizing Data

When we perform a statistical experiment, doing something that generates a piece of data, we are collecting information. If we only perform the experiment once, we have only one piece of data. What we would probably do is to repeat the experiment several times to see if we are getting the same results or if there are
differences from trial to trial. As we keep repeating the experiment, we see that the amount of data that we have keeps increasing in size. The problem we have is how do we manage or organize the data to make the information meaningful to ourselves and to others.

One of the easiest ways to organize data is by means of a frequency distribution. A frequency distribution is a table that pairs classes, the attributes or the numbers, and their frequency. Frequency is the number of times that an experiment results in a particular class as an outcome. The classes can be either qualitative or quantitative. If the classes are qualitative then they will be either the names or the labels or the attributes that we were looking for in the experiment. If the classes are quantitative they can be either discrete values or continuous values. Sometimes when we have a lot of numerical data we may want to group some of the classes together and form a grouped frequency distribution. Let’s look at each one of these types of distributions.

Section 7.1.1 Qualitative Frequency Distribution

EXAMPLE:

You are taking Introduction to College Mathematics and wonder how other people in your program have done who have taken the course from the same instructor. You ask 50 people what they got and record the following letter grades:

B  B  A  D  D  C  BC  AB  D  B
C  AB  AB  BC  C  B  AB  B  A  C
AB  BC  C  B  BC  AB  BC  BC  BC  BC
D  C  D  B  BC  B  C  B  BC  B
BC  B  B  B  BC  B  BC  C  C  C

What we would like to do is to form a frequency distribution based upon the grade received. To form a qualitative distribution we do the following:

1. List each name or attribute
2. Make a tally mark next to each name as we record that data value
3. Total the tally marks to get the frequency

In the above example, we want to create a table with 3 columns. The first column would contain the list of possible grades in some reasonable order: in this case from highest to lowest. The second column is where we do our tallying. Every letter grade in our data gets one tally mark next to the corresponding grade. The third column is the frequency or the number of tally marks that occur for a particular grade. For this problem our chart when complete should look like this.

<table>
<thead>
<tr>
<th>Grades</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BC</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A good way to check that all of the data has been included in the table is to add the frequencies together and see if the total is the same as the number of data values we had to begin with. In this case we have a total of 50 which is how many data values with which we began.

Section 7.1.2 Quantitative Ungrouped Frequency Distribution

Quantitative frequency distributions can be created for either discrete data or continuous data. We also have the options of creating either grouped or ungrouped frequency distributions for quantitative data. An ungrouped quantitative frequency distribution is a frequency distribution in which every data value is its own class or result of the experiment. We would use an ungrouped distribution if the number of outcomes or data values that are possible is relatively small, the number of outcomes is limited, not more than 10 or so. Let’s look at an example of an ungrouped frequency distribution.

**EXAMPLE:**

Traffic Control has been monitoring the number of cars that cross the intersection of Burgoyne and Washington during non-rush hour periods during the day. Over the last several days they have recorded the following 140 numbers of cars crossing that intersection during non-rush hour daylight hour intervals.

| 20 | 19 | 20 | 23 | 23 | 17 | 16 | 19 | 20 | 22 | 22 | 16 | 17 | 20 |
| 19 | 21 | 17 | 23 | 24 | 25 | 21 | 16 | 25 | 24 | 22 | 19 | 21 | 23 |
| 23 | 25 | 19 | 23 | 24 | 23 | 22 | 21 | 23 | 21 | 20 | 25 | 17 | 23 |
| 15 | 19 | 16 | 25 | 25 | 21 | 23 | 17 | 16 | 17 | 23 | 22 | 17 | 24 |
| 19 | 23 | 17 | 25 | 19 | 22 | 17 | 24 | 25 | 22 | 16 | 17 | 23 | 16 |
| 17 | 17 | 24 | 20 | 16 | 20 | 18 | 20 | 19 | 18 | 18 | 16 | 23 | 25 |
| 24 | 16 | 21 | 19 | 16 | 23 | 23 | 20 | 16 | 21 | 21 | 17 | 15 | 21 |
| 17 | 22 | 22 | 23 | 18 | 25 | 20 | 24 | 15 | 24 | 24 | 25 | 17 | 20 |
| 21 | 23 | 21 | 24 | 19 | 22 | 24 | 20 | 18 | 23 | 15 | 19 | 17 | 22 |
| 21 | 22 | 22 | 24 | 19 | 24 | 15 | 20 | 15 | 20 | 20 | 25 | 20 | 24 |

To get a handle on this information we will form an ungrouped frequency distribution. To form this distribution we must do the following:

1. Determine the low and the high value
2. List each data value in order from low to high or from high to low
3. Make a tally mark next to each number in our list for each time it appears in the data
4. Total the tally marks to get the frequency

If you noticed, this is a lot like what we did when we formed a qualitative distribution. The difference is that instead of labels we are using numbers. We are going to create a table with 3 columns. The first column would contain the number of cars per hour reported listed in a reasonable order: in this case from lowest to highest. The second column would be where we do our tallying. Every number of cars per hour in our data set gets one tally mark next to the number of cars per hour possible. The third column is the frequency or the number of tally marks that occur for a particular number of cars per hour. For this problem our chart when complete should look like this.
A good way to check that all of the data has been included in the table is to add the frequencies together and see if the total is the same as the number of data values we had to begin with. In this case we have a total of 140 for the sum of the frequencies which is how many data values with which we began.

Section 7.1.3 Quantitative Grouped Frequency Distribution

As we said above quantitative frequency distributions can be created for either discrete data or continuous data and we also have the options of creating either grouped or ungrouped frequency distributions for quantitative data. A grouped quantitative frequency distribution is a frequency distribution in which several data values or a range of data values form a class. We would use a grouped distribution if the number of outcomes or data values that are possible is large, more than 10 or so. Before we look at an example, we have to know about some guidelines to structure the classes that will be used in the distribution.

Here are just a couple on notes about classes before we look at how to construct the distribution itself:

- There should be between approximately 5 and 20 classes.
- If possible the class width, the difference of the upper end of the class and the lower end of the class, should be an odd number because this will guarantee that the class midpoints are whole numbers instead of decimals or fractions.
- All classes should be the same width.
- The classes must be mutually exclusive. This means that no data value can fall into two different classes.
- The classes must be all inclusive or exhaustive. This means that all data values must be included. The classes must be continuous; that is, there are no gaps in a frequency distribution. Classes that have no values in them must be included unless it's the first or last class in which that class is dropped.

We can now create a grouped frequency distribution by following the below steps:

1. Select an appropriate number of classes to ensure that the distribution will contain between approximately 5 and 20 classes.
2. Determine the largest and the smallest data values in the data set.
3. Determine the range. The range is the difference of the largest minus the smallest data value.
4. Determine the class width. The class width is the range divided by number of classes. Class width should be a whole number, if possible. If the division resulted in a fraction we can round up to the next whole number. If the division resulted in a whole number we will have to add an extra class.
5. Pick a starting point that is less than or equal to the minimum value, the lowest number in the data set. This is the lower limit of the first class. To determine the next lower limit, add the class width to the

<table>
<thead>
<tr>
<th>Cars / hour</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
previous lower limit and keep going until you have a number bigger than the largest value in the data set.

6. To determine the upper limits of the classes we first list in a column all of the lower limits. We then look at the last digit of the number. If our data is whole numbers then the last digit is units and we subtract 1 from the next lower limit to determine the previous upper limit. If our data is decimal we subtract the correct place with a one from the next lower limit to determine the previous upper limit and keep going until you have a number bigger than the maximum value in the data set. Delete any classes above an upper limit that is bigger than the maximum value in the data.

7. Determine the class boundaries. A boundary is point halfway between the upper limit of a class and the lower limit of the next class. The boundaries ensure that no data value can be in two classes. Sometimes it is not necessary to determine the boundaries.

8. Tally the data by determining into which class each data value belongs.

9. Determine the frequencies by counting the tally marks.

Let us now look at an example of how to construct a grouped frequency distribution. We will use the Traffic Control example from above to compare how a grouped frequency distribution looks compared to an ungrouped distribution.

**EXAMPLE:**

Traffic Control has been monitoring the number of cars that cross the intersection of Burgoyne and Washington during non-rush hour periods during the day. Over the last several days they have recorded the following 140 numbers of cars crossing that intersection during non-rush hour daylight hour intervals.

| 20 | 19 | 20 | 23 | 23 | 17 | 16 | 19 | 20 | 22 | 22 | 16 | 17 | 20 |
| 19 | 21 | 17 | 23 | 24 | 25 | 21 | 16 | 25 | 24 | 22 | 19 | 21 | 23 |
| 23 | 25 | 19 | 23 | 24 | 23 | 22 | 21 | 23 | 21 | 20 | 25 | 17 | 23 |
| 15 | 19 | 16 | 25 | 25 | 21 | 23 | 17 | 16 | 17 | 23 | 22 | 17 | 24 |
| 19 | 23 | 17 | 25 | 19 | 22 | 17 | 24 | 25 | 22 | 16 | 17 | 23 | 16 |
| 17 | 17 | 24 | 20 | 16 | 20 | 18 | 20 | 19 | 18 | 18 | 16 | 23 | 25 |
| 24 | 16 | 21 | 19 | 16 | 23 | 23 | 20 | 16 | 21 | 21 | 17 | 15 | 21 |
| 17 | 22 | 22 | 23 | 18 | 25 | 20 | 24 | 15 | 24 | 24 | 25 | 17 | 20 |
| 21 | 23 | 21 | 24 | 19 | 22 | 24 | 20 | 18 | 23 | 15 | 19 | 17 | 22 |
| 21 | 22 | 22 | 24 | 19 | 24 | 15 | 20 | 15 | 20 | 20 | 25 | 20 | 24 |

To form the distribution we will follow the above steps.

Step 1. Select the number of classes.
   Let us choose 5 classes.

Step 2. Determine the high and the low data values.
   The highest value is 25 and the lowest value is 15.

Step 3. Determine the range.
   The range is $25 - 15 = 10$.

Step 4. Determine the class width.
   The class width is $10 \div 5 = 2$. We will have to add a class.

Step 5. Pick a starting point: the first lower limit.
   We will start at 15, the lowest value and our first lower limit.
The rest of the lower limits are 17, 19, 21, 23, 25, and 27. Wait. That gives us six classes. That means that 5 classes will not work with a width of 2 so we need that extra class.

Step 6. Determine upper limits. We list the lower limits in a column.

<table>
<thead>
<tr>
<th>Lower Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
</tr>
<tr>
<td>17</td>
</tr>
<tr>
<td>19</td>
</tr>
<tr>
<td>21</td>
</tr>
<tr>
<td>23</td>
</tr>
<tr>
<td>25</td>
</tr>
<tr>
<td>27</td>
</tr>
</tbody>
</table>

We note that we have whole numbers so we subtract 1 from the next lower limit to get the previous upper limit. Doing that gives us the following list of classes with lower and upper limits.

<table>
<thead>
<tr>
<th>Lower Limit</th>
<th>Upper Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 - 16</td>
<td>16</td>
</tr>
<tr>
<td>17 - 18</td>
<td>18</td>
</tr>
<tr>
<td>19 - 20</td>
<td>20</td>
</tr>
<tr>
<td>21 - 22</td>
<td>22</td>
</tr>
<tr>
<td>23 - 24</td>
<td>24</td>
</tr>
<tr>
<td>25 – 26</td>
<td>26</td>
</tr>
</tbody>
</table>

We can delete the 27 lower limit because we will have no data values that fall in that class.

Step 7. Determine the class boundaries.
Because the data is discrete we do not need boundaries, but if we did they would be the halfway point between an upper limit and the next lower limit. In this case the boundaries are 14.5, 16.5, 18.5, 20.5, 22.5, 24.5, and 26.5.

Step 8 and Step 9 Tally and determine frequencies
We create the following chart.

<table>
<thead>
<tr>
<th>Cars / hour</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 – 16</td>
<td></td>
<td>18</td>
</tr>
<tr>
<td>17 – 18</td>
<td></td>
<td>21</td>
</tr>
<tr>
<td>19 – 20</td>
<td></td>
<td>29</td>
</tr>
<tr>
<td>21 – 22</td>
<td></td>
<td>26</td>
</tr>
<tr>
<td>23 – 24</td>
<td></td>
<td>34</td>
</tr>
<tr>
<td>25 - 26</td>
<td></td>
<td>12</td>
</tr>
</tbody>
</table>

We have created a grouped frequency distribution. Compare this to the ungrouped frequency distribution example above.

We will now form another grouped frequency distribution that is a little more involved because the data we collected has a decimal place in it. The process we will follow is as listed in the above 9 steps.
EXAMPLE:

A student nurse has taken the temperature of 108 people that have visited the clinic in which she works. Their temperatures are listed below. Create a grouped frequency distribution for the temperatures using 7 classes.

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>100.1</td>
<td>103.3</td>
</tr>
<tr>
<td>97.5</td>
<td>100.0</td>
</tr>
<tr>
<td>101.5</td>
<td>101.8</td>
</tr>
<tr>
<td>100.2</td>
<td>102.1</td>
</tr>
<tr>
<td>98.5</td>
<td></td>
</tr>
<tr>
<td>99.1</td>
<td>98.7</td>
</tr>
<tr>
<td>100.6</td>
<td>99.0</td>
</tr>
<tr>
<td>97.5</td>
<td>99.2</td>
</tr>
<tr>
<td>99.1</td>
<td>101.5</td>
</tr>
<tr>
<td>103.0</td>
<td></td>
</tr>
<tr>
<td>100.9</td>
<td>99.3</td>
</tr>
<tr>
<td>103.4</td>
<td>99.9</td>
</tr>
<tr>
<td>98.8</td>
<td>97.1</td>
</tr>
<tr>
<td>104.0</td>
<td>101.1</td>
</tr>
<tr>
<td>98.0</td>
<td></td>
</tr>
<tr>
<td>100.8</td>
<td>97.8</td>
</tr>
<tr>
<td>103.4</td>
<td>101.1</td>
</tr>
<tr>
<td>102.6</td>
<td>99.8</td>
</tr>
<tr>
<td>102.9</td>
<td>103.1</td>
</tr>
<tr>
<td>103.7</td>
<td></td>
</tr>
<tr>
<td>100.0</td>
<td>103.0</td>
</tr>
<tr>
<td>98.7</td>
<td>100.4</td>
</tr>
<tr>
<td>99.1</td>
<td>103.2</td>
</tr>
<tr>
<td>103.5</td>
<td>103.7</td>
</tr>
<tr>
<td>103.4</td>
<td>103.0</td>
</tr>
<tr>
<td>98.2</td>
<td>99.7</td>
</tr>
<tr>
<td>103.4</td>
<td>103.6</td>
</tr>
<tr>
<td>98.7</td>
<td>97.1</td>
</tr>
<tr>
<td>98.2</td>
<td>97.8</td>
</tr>
<tr>
<td>102.9</td>
<td>97.0</td>
</tr>
<tr>
<td>102.9</td>
<td>103.8</td>
</tr>
<tr>
<td>100.2</td>
<td>101.8</td>
</tr>
<tr>
<td>100.4</td>
<td>99.2</td>
</tr>
<tr>
<td>101.6</td>
<td>100.9</td>
</tr>
<tr>
<td>97.3</td>
<td>97.1</td>
</tr>
<tr>
<td>97.8</td>
<td>102.6</td>
</tr>
<tr>
<td>100.7</td>
<td>97.8</td>
</tr>
<tr>
<td>98.8</td>
<td>103.2</td>
</tr>
<tr>
<td>98.7</td>
<td></td>
</tr>
<tr>
<td>102.5</td>
<td>97.8</td>
</tr>
<tr>
<td>99.5</td>
<td>97.0</td>
</tr>
<tr>
<td>103.2</td>
<td>100.6</td>
</tr>
<tr>
<td>99.3</td>
<td>97.87</td>
</tr>
<tr>
<td>102.5</td>
<td></td>
</tr>
<tr>
<td>97.6</td>
<td>98.4</td>
</tr>
<tr>
<td>103.1</td>
<td>97.9</td>
</tr>
<tr>
<td>99.4</td>
<td>99.3</td>
</tr>
<tr>
<td>103.1</td>
<td>103.4</td>
</tr>
<tr>
<td>102.4</td>
<td></td>
</tr>
<tr>
<td>98.0</td>
<td>102.1</td>
</tr>
<tr>
<td>103.9</td>
<td>98.7</td>
</tr>
<tr>
<td>97.1</td>
<td>100.6</td>
</tr>
<tr>
<td>100.9</td>
<td>103.4</td>
</tr>
<tr>
<td>100.0</td>
<td></td>
</tr>
</tbody>
</table>

To form the distribution we will follow the above 9 steps outlined above.

Step 1. Select the number of classes.
   We were asked to use 7 classes.
Step 2. Determine high and low.
   The highest value is 104.0 and the lowest value is 97.0.
Step 3. Determine the range.
   The range is 104.0 – 97.0 = 7.
Step 4. Determine the class width.
   The class width is 7 ÷ 7 = 1. A whole number means we have to add a class.
Step 5. Pick a starting point.
   We will start at 97.0, the lowest value and our first lower limit. The rest of the lower limits are 98.0, 99.0, 100.0, 101.0, 102.0, and 103.0.
   We list the lower limits in a column.
   97.0
   98.0
   99.0
   100.0
   101.0
   102.0
   103.0
   104.0
   105.0
   We note that we have a tenth of a unit so we subtract 0.1 from the next lower limit to get the previous upper limit. Doing that gives us the classes
   97.0 – 97.9
   98.0 – 98.9
   99.0 – 99.9
   100.0 – 100.9
   100.0 – 100.9
We can delete the 105.0 class because we will have no data values that fall in that class.

Step 7. Determine the class boundaries.

Because the data is continuous we may need boundaries. The boundaries are the halfway points between an upper limit and the next lower limit. In this case the boundaries are 96.95, 97.95, 98.95, 100.95, 101.95, 102.95, and 103.9.

Step 8 and Step 9 Tally and determine frequencies

These are given in the completed chart below.

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>97.0 – 97.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>98.0 – 98.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>99.0 – 99.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0 – 100.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>101.0 – 101.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>102.0 – 102.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>103.0 – 103.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>104.0 – 104.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Section 4.1.4 Cumulative Frequency Distribution

A cumulative frequency distribution is a distribution that accumulates, adds-up the frequencies, up to and including a specific class. This type of distribution works well when the data can be ordered in some meaningful way. It is particularly useful when the data is numerical or when the data occurs over time. To create a cumulative frequency distribution we must first create a frequency distribution and then we add a new column. The new column is the cumulative frequencies. The first entry in the cumulative column is the frequency for the first class. The second entry in the cumulative column is the total of the first two frequencies. The third entry in the cumulative column is the total of the first three frequencies. We keep going until we have added together all of the frequencies. This number should check with the number of data values we began with when we started the problem.

EXAMPLE:

In the last example we created a grouped frequency distribution for the temperatures of 108 people who visited a clinic and had their temperature taken by a student nurse. What we would now like to do is to create a cumulative frequency distribution for that data set. We can use what we have created in the last example by adding a new column to the end of the previous table called cumulative frequency. The first row of the table is the same as frequency. The second row is the sum of the first and second frequencies. The third row is the sum of the first three frequencies and so on until we get to the end of the table.
The last entry in the cumulative frequency column should be the total number of data items that we began with when we started the problem.

**Section 7.1.5 Relative Frequency Distribution**

A relative frequency distribution is a frequency distribution that shows the relative frequency of items in each of several non-overlapping classes. The relative frequency is the fraction or proportion of the total number of items belonging to a class and it is generally expresses as a percentage. This definition is applicable to both quantitative and qualitative data. You may wonder why we would need another distribution that is basically the same as a frequency distribution. The relative frequency distribution allows us to compare classes of data that may have been gotten from different sources or collected from different places. For example if we are comparing the number of crimes committed in Madison, Wisconsin, population 208,000, to Madison, Minnesota, population 1750, we would note that the Wisconsin Madison has much more crime in all categories. However, when we look at the percentages, relative frequencies, for certain crimes they are about the same for both places.

To create a relative frequency distribution for a given data set. We first create the frequency distribution for the data. This distribution may be either grouped or ungrouped. We then take the frequency for each class and divide it by the total number of data set items. We should get a decimal answer which we can write to two decimal places. We list these decimals in a new column called relative frequency. The relative frequencies should all add up to 1.0; however, because we may have done some rounding we might be a little lower than one or a little above one.

**EXAMPLE:**

In the last example we created a grouped frequency distribution for the temperatures of 108 people who visited a clinic and had their temperature taken by a student nurse. What we would now like to do is to create a cumulative frequency distribution for that data set. We can use what we have created in that example by adding a new column to the end of the previous table called relative frequency. To determine the relative frequency for each class we take the frequency and divide it by the total. We do this for each class. For example the first class has a frequency of 17. We divide that by 108 to get 0.1574. We round to 0.16 and this is the relative frequency.
### Your Turn!!

1. Covance was performing one of their medical studies and put out a call for health people who were between the ages of 18 and 40. The study would consist of a 5-day stay at their facility for which the participant would get $800. The ages of the 107 people who applied to be in the study are given below.

   22 23 22 19 18 20 27 19 19 24 35 25 25 35
   21 33 19 24 18 22 30 20 29 18 31 28 27 21
   30 27 32 33 23 29 25 31 25 28 30 33 28 22
   21 30 22 22 20 19 33 31 26 18 23 28 19 21
   27 29 38 29 22 22 26 32 25 21 25 24 25 35
   23 31 29 26 32 32 21 20 24 21 28 27 28
   29 19 24 29 25 27 25 21 19 27 32 24 32 19
   18 27 34 25 18 32 31 26 23

   a. Form an ungrouped frequency distribution for the ages given above
   b. Form a grouped frequency distribution for the ages above using 7 classes.
   c. Which distribution is more meaningful; that is, provides more or better information?

2. A local garbage contractor is interested in determining what the distribution of weights of garbage contained in one bag of garbage left at the curb for pickup. He randomly selects one crew to record the weight of the bags of trash before being tossed in the truck. The crew weighs 62 bags of garbage along its route.

   10.8 20.0 27.6 38.1 27.9 21.9 21.8 49.3 33.3 35.5 44.4
   45.2 33.1 10.4 44.4 24.1 37.6 39.0 17.2 31.6 23.1 27.6
   31.0 4.1 23.2 43.0 51.9 20.2 36.8 19.0 15.4 20.7 17.0
   45.4 49.4 18.1 17.4 23.7 19.1 21.0 21.2 29.1 15.0 4.4
   24.8 20.8 25.8 14.6 28.4 27.2 54.5 13.1 45.8 34.8 37.3
   51.6 24.8 26.2 11.3 28.2 11.1 11.7

   a. Create a grouped frequency distribution with 8 classes.
   b. Create a cumulative frequency distribution
3. A 100 gram bag of M & M plain candy contained 115 pieces of candy in the following colors: red (R), orange (O), yellow (Y), brown (N), blue (B), and green (G). The candies came out of the bag in the following order.

   Y O G G R Y O R N B B R R Y
   N R B N Y B Y Y Y Y R N B N
   R N Y O R R G B Y B N Y O B
   R R Y R N N N G R B B B R N
   N N Y N Y N Y B B O N G N Y
   B R Y G N N B O N N R R R R
   O N R N G Y Y N O G N Y R Y
   N N R N N Y N N Y N N O N Y
   Y Y R

   a. Construct a frequency distribution based upon the color of the candy.
   b. Construct a relative frequency distribution for the color of the candy.

4. In the last 76 years the academy award for best actress has been given to actresses of all ages. The ages of those actresses are given below.

   22 37 28 63 32 26 31 27 27 28 30 26 29 24
   38 25 29 41 30 35 35 33 29 38 54 24 25 46
   41 28 40 39 29 27 31 38 29 25 35 60 43 35
   34 34 27 37 42 41 36 32 41 33 31 74 33 50
   38 61 21 41 26 80 42 29 33 35 45 49 39 34
   26 25 33 35 35 28

   a. Create a grouped frequency distribution of the data using 6 classes.
   b. Create a cumulative frequency distribution for the data.
   c. Create a relative frequency distribution for the data.

Section 7.2 Graphing

By now you are wondering why we are doing all this organizational stuff with the data. If you look back at the definition of statistics we have so far only collected and organized the data or information. The next step is to analyze and present the information to someone. We are going to switch the order of these last two items and talk about presentation next.

The most commonly used method used present data is by means of graph or a chart. Creating charts and graph is not difficult. Once we have organized the information, the construction of the visual display is fairly straightforward. The hard part is to decide upon the best method of presentation. Some of the methods available to us are bar charts, line charts, and circle charts.

Section 7.2.1 Bar Charts

Every bar chart consists of two axes: a horizontal axis and a vertical axis. In standard practice, the horizontal axis is where we list the categories or classes of the data and the vertical axis is where we list the frequency,
cumulative frequency, relative frequency or some other numerical value. There are two different kinds of bar charts: a bar chart and a histogram.

A bar chart is a graphical representation of data that is either categorical in nature or discrete, an ungrouped distribution. One of the easiest categorical bar chart to construct is one in which we have some quantity (where an amount is represented by the size of a numerical value) plotted against a category or quality or name. In general, the categories can be presented in any order. A common orderings is to list the categories alphabetically. Another common ordering is to list the categories based upon the ordering of the numerical value. However you decide to arrange the data just be consistent.

**EXAMPLE:**

The table below gives the boiling point at atmospheric pressure for eight different liquids. The boiling points are expressed both in degrees Celsius and in degrees Kelvin (i.e., absolute temperature). We can consider the information to be a categorical distribution where the name of the liquid is the category and the boiling temperature is the frequency.

<table>
<thead>
<tr>
<th>Liquid</th>
<th>Boiling Pt degrees</th>
<th>Boiling Pt degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>acetone</td>
<td>56.20</td>
<td>329.36</td>
</tr>
<tr>
<td>ammonia</td>
<td>-33.35</td>
<td>239.81</td>
</tr>
<tr>
<td>benzene</td>
<td>80.10</td>
<td>353.26</td>
</tr>
<tr>
<td>bromine</td>
<td>58.78</td>
<td>331.94</td>
</tr>
<tr>
<td>ethyl alcohol</td>
<td>78.50</td>
<td>351.66</td>
</tr>
<tr>
<td>isopropyl alcohol</td>
<td>82.40</td>
<td>355.56</td>
</tr>
<tr>
<td>methyl alcohol</td>
<td>64.96</td>
<td>338.12</td>
</tr>
<tr>
<td>water</td>
<td>100.00</td>
<td>373.16</td>
</tr>
</tbody>
</table>

We will use the Celsius data to construct the bar chart. On the horizontal axis we will list the names of the liquids and on the vertical axis we will list boiling temperatures in Celsius. Above each name we will construct a bar from the vertical axis up to the level where the temperature is located. We do this for each pair of data and the result is the above chart. If we had used the absolute temperatures we would have gotten a less exaggerated view of the same data. The absolute temperature chart is given below and is constructed the same
way: the horizontal axis has the names of the liquids and the vertical axis lists boiling temperatures in absolute temperature.

<table>
<thead>
<tr>
<th>Boiling Pt degrees K</th>
</tr>
</thead>
<tbody>
<tr>
<td>acetone</td>
</tr>
<tr>
<td>0.00</td>
</tr>
</tbody>
</table>

Which chart is most appropriate depends upon what information we wish to emphasize. The differences between boiling points are easier to see on the first chart, while the second gives a more accurate comparison of the different liquids.

**EXAMPLE:**

Let say that what we were trying to determine how much blood of a specific type should be kept on hand so that if a transfusion were required the proper type of blood would be available. We will ignore some of the other factors that go into blood typing and just focus on the four major categories of A, B, AB and O. We type the blood of 50 randomly selected individuals and get the following results:

A  B  B  AB  O  O  O  B  AB  B
B  B  O  A  O  A  O  O  O  AB
AB  A  O  B  A  A  O  B  A  AB
B  O  B  O  A  B  B  O  O  O
AB  AB  A  O  B  O  B  O  AB  A

We would like to display our results using a bar chart. The first thing that we must do is to create a frequency distribution of the data. While we are at it, let us also create the relative frequency distribution for this information. The results are given below.

<table>
<thead>
<tr>
<th>Type</th>
<th>Tally</th>
<th>Frequency</th>
<th>Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now that we have the results we can create the bar chart. We have four categories, one for each blood type so we will have four bars on the horizontal axis. We look at the frequencies and not they go as high as 18. So the upper limit of the frequency axis, the vertical, should be 20. Each bar should be the same width. We get the below graph.
If we had wanted to use the relative frequencies our graph would have looked very much like what we have above. The big difference is that instead of the frequency we would have relative frequency on the vertical axis as either percents or as decimals. In the below chart we use percents and tell the viewer that the frequencies are percents.

We can also use a bar chart to represent distributions of discrete numerical data. We do not want too many bars because if we have more than 7 or so the graph will begin to look cluttered. Also we do not want a lot of zero frequencies. If we do have a lot of zero frequencies we will have large gaps between the bars. Consider the following example of discrete numerical data being presented by means of a bar chart.
EXAMPLE:

In 1999, sports utility vehicles averaged between 12 and 19 miles per gallon of gas. A survey was sent to 60 families who had sports utility vehicles and they were asked to compute the miles per gallon, rounded to the nearest mile, they got on their vehicle for a one month period. The results are given below.

12 17 12 14 16 18 12 17 15 15 16 12 15 16 16 12 14 15 12 15 15 19 13 16 18
16 14 12 16 15 12 19 17 18 16 14 13 12 12
12 15 16 19 14 16 15 12 18 16 17 16 15
16 18 15 16 15

To draw the graph we must first create the frequency distribution. It is given below.

<table>
<thead>
<tr>
<th>MPG</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now we can draw the bar chart because our categories are discrete numbers. The graph is given below.
Section 7.2.2 Histograms

A histogram is a type of bar chart in which the bars touch. It is used when we are trying to display continuous numerical data. The reason the bars touch is that one class of data transitions into the next class of data. The graph is drawn similarly to a bar chart: the horizontal axis represents the classes and the vertical axis is the frequency. On the horizontal axis we indicate either the midpoint of the class, under the bar, or we indicate the class limits at the edge of each bar beginning with the lowest class and going up to the end of the upper class.

Let’s look at an example of a histogram,

EXAMPLE:

During the 1998 baseball season Mark McGuire hit 70 home runs. The distance in feet of each home run is given below. Create a histogram to display the data.

<table>
<thead>
<tr>
<th>Distance</th>
<th>Tally</th>
</tr>
</thead>
<tbody>
<tr>
<td>306 – 336</td>
<td></td>
</tr>
<tr>
<td>337 – 367</td>
<td></td>
</tr>
<tr>
<td>368 – 398</td>
<td></td>
</tr>
<tr>
<td>399 – 429</td>
<td></td>
</tr>
<tr>
<td>430 – 460</td>
<td></td>
</tr>
<tr>
<td>461 – 491</td>
<td></td>
</tr>
<tr>
<td>492 – 522</td>
<td></td>
</tr>
<tr>
<td>523 – 553</td>
<td></td>
</tr>
</tbody>
</table>

To create the graph we must first create the frequency distribution. We will use 8 classes. The longest home run was 550 feet and the shortest was 306. The difference is 244 feet. We divide by 8 to get 30.1 which is rounded up to 31. The frequency distribution is as follows:

The graph then is formed by bars that touch and the lower limits are listed at the left edge of each bar and the last upper limit is at the right of the last bar. The graph is shown below
Section 7.2.3 Line Charts

A line chart is any graph that is drawn using a set of points, consisting of two variables or data values, where each point is connected to the next point by means of a straight line. Each data value or variable represents a quantity: the first data value or variable represents a location on the horizontal axis and the second data value or variable is a vertical height or measure above the first data value or variable in a point.

**EXAMPLE:**
Data on the electrical resistivity of tungsten versus absolute temperature is given below.

<table>
<thead>
<tr>
<th>Temp Degree K</th>
<th>Resistivity micro ohm cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>5.65</td>
</tr>
<tr>
<td>400</td>
<td>8.06</td>
</tr>
<tr>
<td>500</td>
<td>10.56</td>
</tr>
<tr>
<td>600</td>
<td>13.23</td>
</tr>
<tr>
<td>700</td>
<td>16.09</td>
</tr>
<tr>
<td>800</td>
<td>19.00</td>
</tr>
<tr>
<td>900</td>
<td>21.94</td>
</tr>
<tr>
<td>1000</td>
<td>24.93</td>
</tr>
<tr>
<td>1100</td>
<td>27.94</td>
</tr>
<tr>
<td>1200</td>
<td>30.98</td>
</tr>
<tr>
<td>1300</td>
<td>34.08</td>
</tr>
<tr>
<td>1400</td>
<td>37.19</td>
</tr>
<tr>
<td>1500</td>
<td>40.36</td>
</tr>
<tr>
<td>1600</td>
<td>43.55</td>
</tr>
<tr>
<td>1700</td>
<td>46.78</td>
</tr>
<tr>
<td>1800</td>
<td>50.05</td>
</tr>
<tr>
<td>1900</td>
<td>53.35</td>
</tr>
<tr>
<td>2000</td>
<td>56.67</td>
</tr>
</tbody>
</table>

Each data pair from the table could be represented as a point on a graph. The first data value of the point, Temperature, would be located on the horizontal axis and the second data value of a point, Resistivity, would be the vertical height of the point on the graph.
From this table we construct the following graph by using a two dimensional grid. The horizontal axis is often called the $x$-axis, while the vertical axis is the $y$-axis. The two axes intersect at the point $(0, 0)$, often called the origin. To graph the data we first determine which variable is "$x$", and which is "$y$". Each data point is plotted by moving horizontally from the origin by an amount $x$ and vertically an amount $y$. If $x$ is positive we move to the right and if $x$ is negative we move left from the origin, while if $y$ is positive we move upwards and if $y$ is negative we move downwards from the origin. The graph of the above data is the points plotted below.

We can see from the graph that the data points look like they will form a line. When this occurs we say that the relationship between the two variables is "approximately linear". If we connect the points with a series of straight lines, we can use the graph to estimate values of the resistivity that were never measured. The graph below shows the plotted point with the addition of the connecting lines.

We can use this graph to estimate resistance for temperatures that were not measured. For example, when the temperature is 1050 degrees K, we can estimate a value of about 27 micro ohm cm for the resistivity of tungsten. We do this by determining where a temperature of 1050° K would be on the horizontal axis and then go straight up until we get to the line. At that point we go to the vertical axis and approximate the resistance. In a similar manner we could estimate the temperature required for a given level of resistance. If we wanted to know the temperature at which we would have a resistance of 15 micro-ohm cm we go up the vertical axis until we get to approximately 15. We then go to the line on our graph and at that point we go straight down to the horizontal axis to read about 650° K.
Section 7.2.4 Cumulative Line Charts

Earlier we looked at how to create a cumulative frequency distribution. We did this by first creating a frequency distribution and then for each class adding together the frequencies for all class up to and including that class. We can draw a graph of the accumulated data by means of a line chart. This chart is drawn just like the chart in the previous example. The classes are listed along the horizontal axis and the accumulated frequencies are plotted vertically above each class. The points are then connected with lines. If our classes represent grouped data the point is plotted at the upper limit.

EXAMPLE:

A veterinarian technician reads in the *World Almanac and Book of Facts* that a charging elephant can attain speeds of up to 29 miles per hour. He decides to see if this is true. He saves his money, buys a used radar speed gun, and goes on safari to Africa. While on safari he gets 60 elephants to charge the vehicle in which he has his equipment. The speeds are recorded below. Create an accumulative frequency distribution and draw the graph.

23  23  27  24

The distribution is as follows:

<table>
<thead>
<tr>
<th>Speed</th>
<th>Tally</th>
<th>Frequency</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>21</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>0</td>
<td>58</td>
</tr>
<tr>
<td>31</td>
<td></td>
<td>0</td>
<td>58</td>
</tr>
<tr>
<td>32</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The graph of the cumulative distribution is given below.
Section 7.2.5 Time-Series Line Charts

A time series graph is a graph that plots data over time. The horizontal axis is the time axis. It can be expressed in any interval of time: minutes, hours, days, months, years, and etcetera. The vertical axis is the frequency or value. The points relate the frequency or value to a specific time.

EXAMPLE:

We will plot stock market prices versus time for a particular stock. The stock is increasing in value when the curve moves up going from left to right and is decreasing when the curve moves down from left to right. For example, in the graph shown below, stock prices are increasing on the days in the interval 0 to 7 and the days in the interval 10 to 14. Stock prices are decreasing on the days in the interval 7 to 10. The stock prices "peaked" or had a "local maximum" at day 7, when the price was $2.05. The stock prices "bottomed out" or had a "local minimum" at day 10, when the price was $1.75. A local maximum always happens at a transition from increasing to decreasing. Similarly, a local minimum occurs at a transition from decreasing to increasing.
Section 7.2.6 Equation Line Charts

Graph can also be generated from equations. An equation is a relation between two variables \( x \) and \( y \). In an equation, we usually imagine that \( x \) is the independent or input variable (whose value we control or set), and that \( y \) is the dependent or output variable (what we measure or record). When the curve has \( y \) "going up" as \( x \) moves from left to right, we say that \( y \) is "increasing", while if the value of \( y \) goes down as \( x \) moves from left to right, \( y \) is "decreasing". This follows the same reasoning that we saw in the time-series graph and discussion above.

**EXAMPLE:**

Suppose we have the formula, \( y = 2x^2 - 3 \), if we substitute, "plug-in," various values of \( x \) into this formula results in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 2x^2 - 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.00</td>
<td>15.00</td>
</tr>
<tr>
<td>-2.50</td>
<td>9.50</td>
</tr>
<tr>
<td>-2.00</td>
<td>5.00</td>
</tr>
<tr>
<td>-1.50</td>
<td>1.50</td>
</tr>
<tr>
<td>-1.00</td>
<td>-1.00</td>
</tr>
<tr>
<td>-0.50</td>
<td>-2.50</td>
</tr>
<tr>
<td>0.00</td>
<td>-3.00</td>
</tr>
<tr>
<td>0.50</td>
<td>-2.50</td>
</tr>
<tr>
<td>1.00</td>
<td>-1.00</td>
</tr>
<tr>
<td>1.50</td>
<td>1.50</td>
</tr>
<tr>
<td>2.00</td>
<td>5.00</td>
</tr>
<tr>
<td>2.50</td>
<td>9.50</td>
</tr>
<tr>
<td>3.00</td>
<td>15.00</td>
</tr>
</tbody>
</table>

Using this table we construct the graph shown below.
Notice that our graph results in a series of points. We could have connected each pair of consecutive points with a straight line but it made more sense to connect them with a curved segment so that the graph is smooth. A graph of this type is called a curve. The graph has a local minimum when $x = 0.00$.

Section 7.2.7 Circle Graphs

A circle graph which is sometimes called a pie graph is a circle that is divided into sections or wedges according to the percentages of frequencies in each category of the distribution. We can use pie graphs to help us represent the data in a relative frequency distribution. Pie graphs are used most often with categorical or attributive data.

To construct a pie graph we first must create the relative frequency distribution of percentages of data within each category or class. Then we take each percentage and multiply by $360^\circ$ to determine about how large each wedge or section of the pie is going to be. We then draw the wedge and label it.

EXAMPLE:

The N. Y. Times Almanac in 2002 listed the following percentages of world wide energy use from different sources. Construct a pie graph of the relative frequency distribution.

<table>
<thead>
<tr>
<th>Energy Type</th>
<th>Percentage used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Petroleum</td>
<td>39.8%</td>
</tr>
<tr>
<td>Coal</td>
<td>23.2%</td>
</tr>
<tr>
<td>Dry Natural Gas</td>
<td>22.4%</td>
</tr>
<tr>
<td>Hydroelectric</td>
<td>7.0%</td>
</tr>
<tr>
<td>Nuclear</td>
<td>6.4%</td>
</tr>
<tr>
<td>Other (Wind, Solar, etc)</td>
<td>1.2%</td>
</tr>
</tbody>
</table>

We first change the percentages to decimal and multiply by $360^\circ$ to determine the size of each wedge. Then we can draw the graph.

<table>
<thead>
<tr>
<th>Energy Type</th>
<th>Percentage used</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Petroleum</td>
<td>39.8%</td>
<td>143</td>
</tr>
<tr>
<td>Coal</td>
<td>23.2%</td>
<td>84</td>
</tr>
<tr>
<td>Dry Natural Gas</td>
<td>22.4%</td>
<td>81</td>
</tr>
<tr>
<td>Hydroelectric</td>
<td>7.0%</td>
<td>25</td>
</tr>
<tr>
<td>Nuclear</td>
<td>6.4%</td>
<td>23</td>
</tr>
<tr>
<td>Other (Wind, Solar, etc)</td>
<td>1.2%</td>
<td>4</td>
</tr>
</tbody>
</table>
Energy Usage by Type

Your Turn!!

1. Below is a table which gives the electrical resistivity (good conductors have less resistivity) in micro-ohm-cm of various metallic elements designated by their chemical symbol.

<table>
<thead>
<tr>
<th>Element</th>
<th>Al</th>
<th>Cu</th>
<th>Au</th>
<th>Pb</th>
<th>Ni</th>
<th>Pt</th>
<th>Ag</th>
<th>Zn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistivity</td>
<td>2.65</td>
<td>1.67</td>
<td>2.35</td>
<td>20.64</td>
<td>6.84</td>
<td>10.6</td>
<td>1.59</td>
<td>5.92</td>
</tr>
</tbody>
</table>

a. Construct a bar graph that displays this information.
b. Which metal is the best conductor?
c. Which metal is the poorest conductor?

2. A sporting goods store collected data about what type of balls its customers bought over the course of one weekend day’s worth of selling. Those little cards do provide the store with all sorts of information. In the list below B is a baseball, K is a basketball, F is a football, G is a package of golf balls, S is a soccer ball, and T is a container of tennis balls.

```
F G F F K B K G T S B F T S
G B K G B S T K G G G S S T
T G K B F K F B B G T F K
G G F S K G T F T T T S K T
F S S G
```

a. Construct a bar chart depicting the number of balls sold
b. Construct a pie chart displaying the percentages of balls sold.
3. The Federal Highway Administration collected data to determine what the added cost to operate a vehicle was due to bad road conditions. They interviewed 40 people about their recent repair work and determined that part of that cost could be attributed to the conditions of the roads over which the people drove. The estimated cost added to the vehicle for these 40 people is given below.

\[
\begin{array}{cccccccccccc}
165 & 156 & 136 & 116 & 91 & 168 & 114 & 127 & 90 & 170 & 122 & 113 \\
112 & 187 & 111 & 172 & 135 & 188 & 136 & 147 & 140 & 131 & 171 & 159 \\
165 & 153 & 125 & 179 & 97 & 163 & 208 & 177 & 152 & 141 & 159 & 169 \\
136 & 155 & 85 & 150
\end{array}
\]

a. Construct a histogram, bar chart in which the bars touch, for this data. Use 6 classes.

b. Construct a cumulative line chart for this data.

4. The data below represents the federal minimum hourly wage for the years shown.

\[
\begin{array}{ccccccccccc}
$1.00 & $1.25 & $1.60 & $2.10 & $3.10 & $3.35 & $3.80 & $4.25 & $5.15 & $5.15
\end{array}
\]

a. Draw a times series chart for the federal minimum wage.

b. In what years did the graph have a maximum value for the federal minimum wage?

c. In what 5 year period did the graph increase the most?

5. The following graph gives the speed of a falling object in m/sec as a function of time measured in sec.

a. What is the speed when the time is 3.0 sec?

b. At what time is the speed 50m/sec?

c. If the speed \( V \) is related to the time \( t \) by the equation: \( V = gt \), where \( g \) is the acceleration of gravity. Estimate the value of \( g \) from this graph.

d. Using the value of \( g \) just calculated; estimate the speed when the time is 10.0 sec.

6. Below is shown the graph of \( y \) versus \( x \). Answer the following questions.
a. When \( x = -1 \), estimate the value of \( y \).

b. When \( x = 0 \), estimate the value of \( y \).

c. When \( x = -3 \), estimate the value of \( y \).

d. When \( x = 2 \), estimate the value of \( y \).

e. When \( x = 2.5 \), estimate the value of \( y \).

f. For what intervals of \( x \) values is \( y \) increasing?

g. For what intervals of \( x \) values is \( y \) decreasing?

h. For what values of \( x \) does \( y \) have a local minimum?

i. For what values of \( x \) does \( y \) have a local maximum?

7. Below is shown the graph of velocity (positive means motion to the right, negative means motion to the left) versus time of a car travelling between two stop lights. The velocity is in units of mph and the time is in units of seconds.

a. How fast was the car moving at the first stop light?

b. How fast was the car moving at the second stop light?

c. Estimate the car’s greatest speed.

d. How long after it left the first stop light did the car travel to the right?
8. For a given circuit the amount of applied voltage, \( V \), and the current in amps, \( I \), are related by means of the follows table.

<table>
<thead>
<tr>
<th>( V ) Voltage</th>
<th>4.0</th>
<th>8.0</th>
<th>12.0</th>
<th>16.0</th>
<th>20.0</th>
<th>24.0</th>
<th>28.0</th>
<th>32.0</th>
<th>36.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I ) Current</td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
<td>2.5</td>
<td>3.0</td>
<td>3.5</td>
<td>4.0</td>
<td>4.5</td>
</tr>
</tbody>
</table>

a. Construct a graph of Voltage versus Current from the following table:
b. From your graph estimate \( V \) when \( I \) is 5.0 amps.

9. The cost of electric power is given by the equation \( C = 0.0558E \), where \( E \) is the amount of electrical energy consumed in kilowatt hours (Kwhr).

a. Generate a graph of \( C \) versus \( E \), for \( 0 < E < 1000 \) Kwhr.
b. Estimate the value of \( C \) when \( E \) is 600 Kwhr.
Section 7.3 Descriptive Statistics

Descriptive statistics is the body of methods used to represent and summarize sets of numerical data. Descriptive statistics provides a means of describing how a set of measurements (for example, people’s weights measured in pounds) is distributed. Descriptive statistics involves two different kinds of summary statistics. These summary statistics are called measures of center and measures of spread.

Section 7.3.1 Measures of Center

Measures of center involve a “typical” data value about which all the other data values are distributed. The most commonly used measures of center are the mean, the median and the mode. The mean is sometimes called the arithmetic average and it is the sum of all the data values divided by the number of data values. If the data is arranged in order, high to low or low to high, then the median is the “middle” data value with as many data values above it as below it. The mode is the data value that occurs the most often (has the highest frequency). The mode is only meaningful for large sets of data where it is likely for the same score to be measured more than once.

If we were to graph a large data set and the mean, median, and the mode were all the same, our graph would look like the graph below on the left. A graph of this type is called symmetric: one side of the graph looks just like the other side of the graph. Symmetric graphs result from distributions of data in which the mean, median and mode are all equal. If any of these three are different from the other two or if all three are different the graph will no longer be symmetric and will become skewed. Therefore, any differences between these three measures are an indication of skewness or departure from a symmetric distribution.
Section 7.3.2 Measures of Spread

The most commonly used measures of spread are range, standard deviation, and inter-quartile range. The range is the distance between the highest data value (the maximum) and the lowest data value (the minimum). The standard deviation measures a “typical” distance of data values in the distribution from the mean. The inter-quartile range is the distance covered by the “middle half” of the data from the score that marks the first quarter point to the score that marks the third quarter point.

For the famous “bell-shaped” curve (what is generally called a normal distribution curve or a Gaussian distribution), 68% of all scores are found within one standard deviation of the mean and 95.5% of all scores are within two standard deviations of the mean. More generally, for any distribution of scores, at least 75% of all scores must be within two standard deviations of the mean.
To make the calculation of these statistics easier the following notation will be used.

- \( n \) represents the number of data values which make up the data set or the sample
- \( x_i \) represents a subscripted variable. The index \( i \) can take values from 1 to \( n \) with each value of \( x_i \) being a particular data value in the data set. Furthermore, it will be assumed that the scores have been sorted from smallest to largest (an increasing sort). Thus, \( x_1 \) is the minimum and \( x_n \) is the maximum.
- \( f_i \) represents the frequency for a corresponding \( x_i \)
- \( \Sigma \) is the upper case Greek letter sigma. This symbol stands for the mathematical operation of summation and means we take the sum of the data values: \( x_1 + x_2 + x_3 + \ldots + x_n \). This sum can be more concisely written as \( \sum_{i=1}^{n} x_i \). The variable \( i \) is called a “dummy” summation index and is just used to signify that various values of \( x \) are being summed. The designation \( i = 1 \) beneath the sigma indicates where the sum begins and the \( n \) above the sigma indicates where the sum ends. In statistics it is nearly always the case that sums begin at 1 and end at \( n \), so the following abbreviated symbols are often employed.

\[
\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n = \sum x_i = \sum x_i = \sum x .
\]

Section 7.3.4  Mean Calculations

The mean is the arithmetic average and it is usually indicated by \( \bar{x} \) (called x bar). Sometimes a second symbol for the mean is used, called the population mean, when all possible data values have been measured. The population mean is represented by the Greek letter lower case mu, \( \mu \).

The mean can be computed by adding together all of the data values and dividing by the number of data values that were added together. In terms of a formula, we write the mean calculation as

\[
\bar{x} = \frac{\sum x_i}{n} .
\]

EXAMPLE:
In the years 1992 – 1994 the United States had 18 space shuttle missions. The duration of these missions in days is given below. Compute the mean of the space shuttle missions.

\[
\bar{x} = \frac{\sum x_i}{n}
\]

\[
\bar{x} = \frac{8 + 9 + 9 + 14 + 8 + 8 + 10 + 7 + 6 + 9 + 7 + 8 + 10 + 14 + 11 + 8 + 14 + 11}{18}
\]

\[
\bar{x} = \frac{171}{18}
\]

\[
\bar{x} = 9.5
\]

If there is a lot of data, we may want to form a frequency distribution before we begin to do any computations. The benefits are that the computations we will have to do are reduced because we rely on the fact that multiplication is repeated addition. We do have another formula for the mean that we will use when we have a frequency distribution.

\[
\bar{x} = \frac{\sum x_i \cdot f_i}{\sum f_i}
\]

If the data is ungrouped we use each data value for \(x_i\) in the formula. If the data is grouped then we have to determine the midpoint of each class and use that for \(x_i\). The midpoint is the average of the upper and lower limits of the class.

**EXAMPLE:**

A 15 question practice driver’s education test was administered to a group of people planning on taking the written portion of the driver’s license examination. The number of correct questions a person got is shown below.

<table>
<thead>
<tr>
<th>Correct</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
</tr>
</tbody>
</table>

**Compute the mean number correct.**
We note that in the formula \( \bar{x} = \frac{\sum x_i f_i}{\sum f_i} \) that the numerator requires we sum the product of the \( x \)'s times their corresponding \( f \)'s. To do this we will create a new column to record the products. The denominator requires us to sum all the frequencies so that we can divide the two sums. Our new table looks like

<table>
<thead>
<tr>
<th>Correct</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( f )</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>Total</td>
<td>( \sum f_i = 120 )</td>
</tr>
</tbody>
</table>

We now substitute into the formula and simplify.

\[
\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1103}{120} = 9.19
\]

The mean number correct on the test was 9.19 correct.

**EXAMPLE:**

A copy center has 75 copy machines. At some point each machine required some type of service. The number of days between servicing for the machines is given below in the table. Compute the mean days between required servicing of the machines.

<table>
<thead>
<tr>
<th>Days</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.5 – 18.5</td>
<td>14</td>
</tr>
<tr>
<td>18.5 – 21.5</td>
<td>12</td>
</tr>
<tr>
<td>21.5 – 24.8</td>
<td>18</td>
</tr>
<tr>
<td>24.5 – 27.5</td>
<td>10</td>
</tr>
<tr>
<td>27.5 – 30.5</td>
<td>15</td>
</tr>
<tr>
<td>35.5 – 33.5</td>
<td>6</td>
</tr>
</tbody>
</table>

We must first compute the midpoint of each class. This is done by computing the midpoint of the first class, add the limits and divide by 2, and then add the class to the midpoint to get the next midpoint. Keep adding the width to get each following midpoint.
We now substitute into the formula and simplify.

\[
\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1779}{75} = 23.72
\]

The mean number of days between servicing is 23.72 days.

### Section 7.3.5 Mode Calculations

The mode of a data set is the data value that occurs the most often in the data set. If we have a frequency distribution, the mode is the data value with the largest frequency. If the frequency distribution is a grouped distribution, then we can’t really determine which data value occurs most but we can determine which class occurs most often.

**Example:**

In the years 1992 – 1994 the United States had 18 space shuttle missions. The duration of these missions in days is given below. Determine the mode of the space shuttle missions.

8 9 9 14 8 8 10 7 6
9 7 8 10 14 11 8 14 11

We will form a frequency distribution for the data. The value that occurs most often is the mode. From the table we see that 8 days occurs most often. That is the mode of this data.
Example:

A copy center has 75 copy machines. At some point each machine required some type of service. The number of days between servicing for the machines is given below in the table. Compute the mode of the days between required service for the machines.

<table>
<thead>
<tr>
<th>Days</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.5 – 18.5</td>
<td>14</td>
</tr>
<tr>
<td>18.5 – 21.5</td>
<td>12</td>
</tr>
<tr>
<td>21.5 – 24.8</td>
<td>18</td>
</tr>
<tr>
<td>24.5 – 27.5</td>
<td>10</td>
</tr>
<tr>
<td>27.5 – 30.5</td>
<td>15</td>
</tr>
<tr>
<td>35.5 – 33.5</td>
<td>6</td>
</tr>
</tbody>
</table>

We look at the distribution and see that the class 21.5 – 24.8 has the highest frequency. This is the modal class. Most repairs will occur between 21.5 days and 24.5 days.

Section 7.3.6 Median Calculations

The median of a set of data values is the data value or average of two values that is located in the exact middle of the data set. The median can be calculated by the following procedure.

a. Arrange all of the data values in order from low to high.
b. Determine how many data values, \( n \), are in the data set.
c. Use the formula \( d(m) = 1 + \frac{1}{2}(n - 1) \) to determine the position or depth of the median in the data set.
   i. If \( d(m) \) is a whole number.
      Beginning at the low end of the data, count each data value until you get to \( d(m) \). This is the median. There should be as many data values below the median as above the median.
   ii. If \( d(m) \) is a decimal then \( d(m) \) will be located between two data values.
      Beginning at the low end of the data, count each data value until you get to the whole number part of the decimal \( d(m) \).
      Multiply the difference of the two number either side of where \( d(m) \) fell by the decimal part of \( d(m) \). This will always be 0.5.
      Add this to the number below \( d(m) \). The result is the median. There should be as many data values below the median as above the median.
Example:

It has been observed that time seems to drag on in some classes. This is not one of those classes. To determine if people could really determine when one minute (60 seconds) has passed 13 students separated from other students were asked to indicate when they thought one minute had elapsed. The recorded times for these students is given below. What is the median time from this sample of students?

53 52 75 62 68 58 49 49 65 63 51 64 54

To determine the median we order the data from low to high.

49 49 51 52 53 54 58 62 63 64 65 68 75

There are 13 data values so \( m = 1 + \frac{1}{2}(n - 1) = 1 + \frac{1}{2}(13 - 1) = 1 + \frac{1}{2}(12) = 1 + 6 = 7 \). The median is in the 7th position.

49 49 51 52 53 54 58 62 63 64 65 68 75

The median is 58 as this is in the 7th position of the arranged data.

Example:

Listed below are the intervals in minutes between eruptions of the Old Faithful geyser in Yellowstone National Park. What is the median time between eruptions?

98 92 95 87 96 90 65 92 95 93 98 94

To determine the median we order the data from low to high.

65 87 90 92 92 93 94 95 95 96 98 98

There are 12 data values. We compute \( m = 1 + \frac{1}{2}(n - 1) = 1 + \frac{1}{2}(12 - 1) = 1 + \frac{1}{2}(11) = 1 + 5.5 = 6.5 \). The median is between the 6th and 7th data values.

65 87 90 92 92 93 94 95 95 96 98 98

The difference between 94 and 93 is 1.
The product of 0.5 and 1 is 0.5.
Add 0.5 to the number in the position below the depth of the median which is 93. The sum is 93.5 which is the median. The median time between eruptions is 93.5 minutes.

Medians can also be determined for data that is given to us as a frequency distribution. Most of the process of determining the median will be the same as what we did earlier. The data will already be ordered. We must compute how many data values there are; we do this by summing the frequencies. We compute the depth of the median, \( m = 1 + \frac{1}{2}(\sum f_i - 1) \). Notice that \( n \) was replaced by \( \sum f_i \) but it is still how many data values there are in the data set. We must form the cumulative distribution for the data. Compare the depth of the median.
with the cumulative frequencies beginning with the smallest cumulative frequency. The median class is the first class that has a cumulative frequency larger than the depth of the median.

**EXAMPLE:**

A 15 question practice driver’s education test was administered to a group of people planning on taking the written portion of the driver’s license examination. The number of correct questions a person got is shown below. Determine the median number correct.

<table>
<thead>
<tr>
<th>Correct</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
</tr>
</tbody>
</table>

The data is already ordered so we must create the cumulative frequency distribution. As we are doing this we will also determine \( \sum f_i \) because that should be the last entry in the cumulative column.

<table>
<thead>
<tr>
<th>Correct</th>
<th>Frequency</th>
<th>Cumulative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>24</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>37</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>51</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
<td>67</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
<td>81</td>
</tr>
<tr>
<td>11</td>
<td>9</td>
<td>90</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
<td>108</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
<td>115</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>120</td>
</tr>
</tbody>
</table>

We now compute the depth of the median.

\[
d(m) = 1 + \frac{1}{2}(\sum f_i - 1) = 1 + \frac{1}{2}(120 - 1) = 1 + \frac{1}{2}(119) = 1 + 59.5 = 60.5.
\]

The first class that exceeds the depth of the median is the 9 class with 67. The previous classes only totaled up to 51. The median number correct is 9.
Section 7.3.7  Range Calculations

The range, symbolized by an $R$, is a measure of the spread of the data. The range is the highest data value ($H$) minus the lowest data value ($L$). That is, $R = H - L$. The range of the data can be affected by extreme values.

EXAMPLE:

Listed below are the intervals in minutes between eruptions of the Old Faithful geyser in Yellowstone National Park. What is the range of times between eruptions?

\[ 98 \quad 92 \quad 95 \quad 87 \quad 96 \quad 90 \quad 65 \quad 92 \quad 95 \quad 93 \quad 98 \quad 94 \]

The highest value is 98 and the lowest value is 65. The range is computed as follows: $R = H - L = 98 - 65 = 33$. The range of times is 33 minutes.
Section 7.3.8 Standard Deviation Calculations

Standard deviation is a method statistical tool that allows us to determine how much variability is in the data. If we have a large standard deviation then there is a lot of variability. If there is a small standard deviation then there is little variability. Most manufacturing processes would like to keep variability to a minimum. Consider a group of soft drinks sold in 2-liter bottles on a shelf in the grocery store. Some of them look fuller than the others. This difference is a result of variability which we can measure by means of the standard deviation.

Standard deviation is calculated by the following formula 

\[ s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}. \]

In the formula we use the symbol \( s \) to indicate the sample standard deviation. If all possible data values have been measured, then we have the slightly different formula (which gives a slightly smaller result) because we can compute the population standard deviation

\[ \sigma = \sqrt{\frac{\sum (x_i - \mu)^2}{n}}. \]

The Greek letter \( \sigma \) is the lower case sigma. It is used to designate that this is for a population of data values as opposed to just a sample. Both of these formulas can be manipulated using algebra to give the computationally more efficient formulas shown below.

For a sample standard deviation we would rather use the formula 

\[ s = \sqrt{\frac{\sum x_i^2 - (\sum x_i)^2}{n-1}} \]

and for a population standard deviation the formula

\[ \sigma = \sqrt{\frac{\sum x_i^2 - (\sum x_i)^2}{n}}. \]

The order of operations is crucial. \( \sum x_i^2 \) means that each \( x_i \), a data value, is first squared and then summed (i.e., this is the sum of squares). In contrast, \( (\sum x_i)^2 \) means that the \( x_i \)'s are first summed and then this answer is then squared (i.e., this is the square of the sum).

**EXAMPLE:**

We have a data set that has 5 data values which are 1, 3, 4, 5, and 7.

Then \( n = 5 \) and \( x_1 = 1, x_2 = 3, x_3 = 4, x_4 = 5, \) and \( x_5 = 7 \) and

\[ \sum x_i^2 = 1 + 9 + 16 + 25 + 49 = 100, \]

while

\[ (\sum x_i)^2 = (1 + 3 + 4 + 5 + 7)^2 = 20^2 = 400. \]

If we compute the sample mean of this data set we have \( \bar{x} = \frac{20}{5} = 4 \), and if we compute the sample standard deviation we have

\[ s = \sqrt{\frac{100 - \frac{400}{5}}{5 - 1}} = \sqrt{5} = 2.2361 \]
If we treat this data as population data then the results are that the population mean is \( \mu = \frac{20}{5} = 4 \), and the population standard deviation is \( \sigma = \sqrt{\frac{100 - 400}{5}} = \sqrt{4} = 2 \).

Note that the means are the same but the standard deviations are a little different.

If our data is presented to us as a frequency distribution, the formulas will change a little because we now have frequencies to deal with.

For a sample the standard deviation formula becomes

\[
\sigma = \sqrt{\frac{\sum f_i x_i^2 - (\sum f_i x_i)^2}{\sum f_i} - 1}
\]

and the population standard deviation becomes

\[
\sigma = \sqrt{\frac{\sum f_i x_i^2 - (\sum f_i x_i)^2}{\sum f_i}}.
\]

These formulas may seem extremely complicated but if we arrange the data in a table the computations become easier. Let’s look at a process that should help to make this manageable.

For sample data that we get in a list,
1. List the data in the first column
2. Determine the sum of the first column. This is \( \sum x_i \)
3. In the second column list the squares of the data in the first column
4. Determine the sum of the second column. \( \sum x_i^2 \)
5. Determine how many data values there are. This is \( n \).
6. Substitute into the formula for sample or population and simplify.
EXAMPLE:

In January 10 cities were selected at random and their high temperatures were recorded. The sample is given below. Compute the mean and standard deviation for this data.

\[ \begin{array}{cccccccccc}
50 & 37 & 29 & 54 & 30 & 61 & 47 & 38 & 34 & 61 \\
\end{array} \]

To determine the sample mean we will use the formula \( \bar{x} = \frac{\sum x_i}{n} \) and for the sample standard deviation the formula \( s = \sqrt{\frac{\sum x_i^2 - \left(\frac{\sum x_i}{n}\right)^2}{n-1}} \).

\[
\begin{array}{|c|c|}
\hline
x_i & x_i^2 \\
\hline
50 & 2500 \\
37 & 1369 \\
29 & 841 \\
54 & 2961 \\
30 & 900 \\
61 & 3721 \\
47 & 2209 \\
38 & 1444 \\
34 & 1156 \\
61 & 3721 \\
\hline
\text{Sum} = & 441 & 20777 \\
\hline
\end{array}
\]

In this problem we know that \( n = 10 \).
We compute the mean with the information from the table.
\[
\bar{x} = \frac{\sum x_i}{n} = \frac{441}{10} = 44.1
\]

We compute the sample standard deviation with the information from the table.
For sample data that we get in a frequency distribution, to compute the standard deviation we

2. List the data classes in the first column. If required determine the midpoint of the classes and consider this the first column.

3. In the second column list the frequencies of the classes and determine the sum \( \sum f_i \).

4. In the third column compute the products of the frequencies time the data value (midpoint) and determine the sum of the third column. \( \sum f_i x_i \).

5. In the fourth column determine the squares of the data values (midpoint),

6. In the fifth column determine the products of the frequencies times the data value (midpoint) and determine their sum \( \sum f_i x_i^2 \).

7. Substitute into the formula for sample or population and simplify.

**EXAMPLE:**

Twenty runners who finished a 10 kilometer race were asked the question “How many miles a week do you run? Their responses are summarized in the grouped frequency distribution below. Determine the mean and standard deviation for this sample.
Following the directions given above, we set-up a table of values.

<table>
<thead>
<tr>
<th>Class</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.5 – 10.5</td>
<td>1</td>
</tr>
<tr>
<td>10.5 – 15.5</td>
<td>2</td>
</tr>
<tr>
<td>15.5 – 20.5</td>
<td>3</td>
</tr>
<tr>
<td>20.5 – 25.5</td>
<td>5</td>
</tr>
<tr>
<td>25.5 – 30.5</td>
<td>4</td>
</tr>
<tr>
<td>30.5 – 35.5</td>
<td>3</td>
</tr>
<tr>
<td>35.5 – 40.5</td>
<td>2</td>
</tr>
</tbody>
</table>

We can compute the mean using the formula

$$
\bar{x} = \frac{\sum x_i \cdot f_i}{\sum f_i} = \frac{490}{20} = 24.5
$$

The standard deviation is computed using the formula

$$
\sigma = \sqrt{\frac{\sum x_i^2 \cdot f_i - (\sum x_i \cdot f_i)^2}{\sum f_i}}
$$
A quartile is a position in an arranged, from lowest to highest, data set that has exactly 25%, 50% or 75% of the data below that point. Every data set had three quartile points: a first quartile, a second quartile (called the median) and a third quartile. The quartiles points divide the data set into four parts each containing 25% of the data.

The procedure we use to compute the depth of the median, \( d(m) = 1 + \frac{1}{2}(n-1) \), can be generalized to calculate the first and the third quartiles. We use the symbols \( Q_1 \) and \( Q_3 \) to represent the data values one quarter and three quarters of the way through the data respectively. The depth of \( Q_1 \) (called the first quartile) is given by the formula \( d(Q_1) = 1 + \frac{1}{4}(n-1) \), and the depth of \( Q_3 \) (called the third quartile) is given by the formula \( d(Q_3) = 1 + \frac{3}{4}(n-1) \).

One last statistic that we may have interest in computing is the inter-quartile range. The inter-quartile range tells us the range of the middle 50% of all the data. To compute the inter-quartile range we determine the difference of the third quartile minus the first quartile or \( Q_3 - Q_1 \).

**EXAMPLE:**

Compute the depth of the first and third quartile if \( n = 16 \).

To compute the depth of \( Q_1 \) and \( Q_3 \) we can use the formulas.
Now that we can compute the depth of a quartile, the next step is to compute the actual quartile values. To compute these values we will do just like we did with the median except that there are some slight differences.

To compute quartiles,
1. Arrange all of the data values in order from low to high.
2. Determine how many data values, \( n \), are in the data set.
3. Use the formula to determine the position or depth of the quartile in the data set:
   \[ d(Q_i) = 1 + \frac{1}{4}(n - 1) \quad \text{for} \quad d(Q_1) \]
   \[ d(Q_3) = 1 + \frac{3}{4}(n - 1) \quad \text{for} \quad d(Q_3) \]
   iii. If the depth is a whole number.
      Begin at the low end of the data and count each data value until you get to the \( d(Q_1) \) or \( d(Q_3) \) position. This is the first quartile or third quartile respectively.
   iv. If \( d(Q_1) \) or \( d(Q_3) \) is a decimal then the quartile is located between two data values.
      Beginning at the low end of the data, count each data value until you get to the whole number part of the decimal for the depth.
      Multiply the difference of the two numbers either side of the depth position by the decimal part of the depth. This will be either 0.25 or 0.5 or 0.75
      Add this to the number below the depth position. The result is the appropriate quartile.
EXAMPLE:

You work in an area of a plant that you consider to be extremely noisy and are aware of noise induced hearing loss. This type of loss can occur from 8 hours of exposure at a sound intensity of 85 decibels, a measure of sound pressure against the eardrum. A running gas lawn mower is about 85-80 decibel. Sound levels above 120 decibels, a jackhammer, for a shorter period of time can also cause permanent loss. You complain to the plant’s public safety officer who monitors the sound in your area for 15 days. The peak decibel readings in your area as follows:

100  59  78  97  84  64  53  59  89  88  94  66  57  62  64

Compute the first quartile, median, and third quartile sound levels.

We arrange the data in order to get

53  57  59  59  62  64  64  66  78  84  88  89  94  97  100

There are 15 data values so \( n = 15 \).

We now compute \( d(Q_1) \)

\[
d(Q_1) = 1 + \frac{1}{4}(n-1) = 1 + \frac{1}{4}(15-1) = 1 + \frac{1}{4}(14) = 1 + 3.5 = 4.5
\]

Next we determine \( Q_1 \). In the ordered data set 4.5 places is between 59 and 62. The difference of these two numbers is 3 and 0.5 times 3 is 1.5. We add 1.5 to the lower of the two numbers, 59, and get 60.5. The first quartile is 60.5.

We now compute the median

\[
d(M) = 1 + \frac{1}{2}(n-1) = 1 + \frac{1}{2}(15-1) = 1 + \frac{1}{2}(14) = 1 + 7 = 8
\]

Next we determine the median. In the ordered data set 8 places is 66. The median is 66.

We now compute \( d(Q_3) \)

\[
d(Q_3) = 1 + \frac{3}{4}(n-1) = 1 + \frac{3}{4}(15-1) = 1 + \frac{3}{4}(14) = 1 + 10.5 = 11.5
\]

Next we determine \( Q_3 \). In the ordered data set 11.5 places is between 88 and 89. The difference of these two numbers is 1 and 0.5 times 1 is 0.5. We add 0.5 to the lower of the two numbers, 88, and get 88.5. The third quartile is 88.5.

The inter-quartile range can be easily determined once we know \( Q_3 \) and \( Q_1 \). For this example we take 88.5 – 60.5 = 28.5. The range of the middle 50% of all the data is 28.5
Section 7.3.10  Box Plots

So what does that information tell us about dangerous levels of noise? With a third quartile of 88.5 decibels we are safe 88.5% of the time and are exposed to dangerous levels 25% of the time. We would like to display this information graphically. A graph that displays this information is called a “box plot” or a “box and whiskers plot.” This plot is based upon a 5 number summary. The 5 numbers in the summary are the low data value, the first quartile, the median, the third quartile, and the high data value.

The box plot in its simplest form is a scaled diagram with a straight line (whisker) drawn from the minimum score to \( Q_1 \). From \( Q_1 \) to \( Q_3 \) a box is drawn with the position of the median marked. A second whisker then drawn from \( Q_3 \) to the maximum score. This construction generates a graphical representation of the distribution of the data set.

To construct a 5-number summary so that we can draw a box plot we must

1. Order the data from low to high
2. Determine the low data value
3. Determine the first quartile
4. Determine the median
5. Determine the third quartile
6. determine the high data value
7. Draw the plot.

**EXAMPLE:**

For the decibel problem above draw the box plot of that data. The data was as follow:

\[
100 \ 59 \ 78 \ 97 \ 84 \ 64 \ 53 \ 59 \ 89 \ 88 \ 94 \ 66 \ 57 \ 62 \ 64
\]

Arrange the data from low to high.

\[
53 \ 57 \ 59 \ 59 \ 62 \ 64 \ 64 \ 66 \ 78 \ 84 \ 88 \ 89 \ 94 \ 97 \ 100
\]

Determine the low data value: 53. We got this from looking at the arranged list.
Determine the first quartile: 60.6. We computed this in the last example
Determine the median: 66. We computed this in the last example
Determine the third quartile: 88.5. We computed this in the last example
Determine the high data value: 100. We got this from looking at the arranged list.

Draw the plot:

[Box plot diagram]

Madison College’s College Mathematics Textbook  Page 252 of 256
Your Turn!!

1. A simple maze was constructed to determine the amount of time to the nearest second that it would take a rat to find the food at the end of the maze. Twelve rats were tested. The time it took each rat is given below.

18.1 , 19.7 , 17.8 , 19.1 , 16.5 , 20.0 , 18.5 , 17.9 , 20.9 , 20.3 , 19.4 , 18.3

Compute:
   a. The mean, \( \bar{x} \)
   b. The median
   c. The mode, if one exists
   d. The first quartile, \( Q_1 \)
   e. The third quartile, \( Q_3 \)
   f. The inter-quartile range
   g. The range
   h. The sample standard deviation, \( s \)
   i. The population standard deviation, \( \sigma \)
   j. Draw a box and whiskers plot for this data

2. The following frequency distribution is based on the test scores on the Statistics chapter of Intro to College Math for three sections taught by the same teacher last year.

<table>
<thead>
<tr>
<th>X</th>
<th>f</th>
<th>X</th>
<th>f</th>
<th>X</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>69</td>
<td>1</td>
<td>79</td>
<td>4</td>
<td>89</td>
<td>3</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>80</td>
<td>3</td>
<td>90</td>
<td>0</td>
</tr>
<tr>
<td>71</td>
<td>0</td>
<td>81</td>
<td>4</td>
<td>91</td>
<td>2</td>
</tr>
<tr>
<td>72</td>
<td>2</td>
<td>82</td>
<td>5</td>
<td>92</td>
<td>1</td>
</tr>
<tr>
<td>73</td>
<td>1</td>
<td>83</td>
<td>6</td>
<td>93</td>
<td>0</td>
</tr>
<tr>
<td>74</td>
<td>3</td>
<td>84</td>
<td>7</td>
<td>94</td>
<td>1</td>
</tr>
<tr>
<td>75</td>
<td>1</td>
<td>85</td>
<td>7</td>
<td>95</td>
<td>0</td>
</tr>
<tr>
<td>76</td>
<td>0</td>
<td>86</td>
<td>9</td>
<td>96</td>
<td>1</td>
</tr>
<tr>
<td>77</td>
<td>3</td>
<td>87</td>
<td>4</td>
<td>97</td>
<td>1</td>
</tr>
<tr>
<td>78</td>
<td>2</td>
<td>88</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Compute:
   a. The mean, \( \bar{x} \)
   b. The median
   c. The mode, if one exists
   d. The first quartile, \( Q_1 \)
   e. The third quartile, \( Q_3 \)
   f. The inter-quartile range
   g. The range
   h. The sample standard deviation, \( s \)
   i. The population standard deviation, \( \sigma \)
   j. Draw a box and whiskers plot for this data
Chapter 7 Practice Exam

1. What is statistics?

2. What kind of data is data that is said to be quantitative?

3. During a census, one of the questions you are asked to respond to is “What type of housing do you have?” The choices are apartment (A), condominium (C), House (H), or Mobile Home (M). Fifty people from the town of Happyville responded as follows:

   H C H M H A C A M C
   M C A M A C C M C C
   H A H M H C C C M A C H
   C H M A A H M H A A H
   C C M A C M M

   a. Form a frequency distribution for this data
   b. Create a bar chart for the data.
   c. Create a circle graph for the data.
4. A cashier was bored one day while working at the Quick Stuff Mart. To pass the time the cashier recorded the number of purchases a customer made other than gas. The next 30 people who bought something besides gas at the store purchased the following number of items:

\[ 2 \ 9 \ 4 \ 3 \ 6 \ 6 \ 2 \ 8 \ 6 \ 5 \]
\[ 7 \ 5 \ 3 \ 8 \ 6 \ 6 \ 2 \ 3 \ 2 \ 4 \]
\[ 6 \]
\[ 9 \ 9 \ 8 \ 9 \ 4 \ 2 \ 1 \ 7 \ 4 \]

Compute:

a. The mean, \( \bar{x} \)

b. The median

c. The mode, if one exists

d. The first quartile, \( Q_1 \)

e. The third quartile, \( Q_3 \)

f. The inter-quartile range

h. The sample standard deviation, \( s \)
i. The population standard deviation, \( \sigma \)

j. Draw a bar chart for this data

k. What is the relative frequency for the 7 purchase class

l. Draw a cumulative graph for this data

m. Draw a box and whiskers plot for this data

5. The Pittsburg Tribune Review reported the murder rates 25 selected but unnamed cities in the United States. The murder rates they reported were as follows:

\[ 248 \ 270 \ 366 \ 149 \ 109 \ 348 \ 71 \ 73 \ 68 \]
\[ 598 \ 74 \ 226 \ 241 \ 73 \ 278 \ 514 \ 41 \ 46 \]
\[ 63 \ 69 \ 597 \ 39 \ 34 \ 65 \ 27 \]

Compute:

a. The mean, \( \bar{x} \)

b. The median

c. The mode, if one exists

d. The first quartile, \( Q_1 \)

e. The third quartile, \( Q_3 \)

f. The inter-quartile range

g. The range

h. The sample standard deviation, \( s \)
i. The population standard deviation, \( \sigma \)

j. Draw a box and whiskers plot for this data

k. Create a grouped frequency distribution with 7 classes. Draw a histogram of the data
6. The National Association of Theater Owners has kept track of the number of indoor movie theaters in the United States since 1987. The year and number of theaters is given below.

<table>
<thead>
<tr>
<th>Year</th>
<th>Number</th>
<th>Year</th>
<th>Number</th>
<th>Year</th>
<th>Number</th>
<th>Year</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>22904</td>
<td>1996</td>
<td>28905</td>
<td>2002</td>
<td>35170</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1991</td>
<td>23740</td>
<td>1997</td>
<td>31050</td>
<td>2003</td>
<td>35361</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1992</td>
<td>24344</td>
<td>1998</td>
<td>33418</td>
<td>2004</td>
<td>36012</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Create a time series graph of the data
b. Does the graph reveal any trends in the data?
c. Have there been any years in which the number of theaters declined? What years?

7. A truck rental company charges customers a flat fee of $95 a day for use of a truck and $0.45 per mile driven after the first 100 miles.

   a. What is the cost if you rent the truck for 1 day and drive 75 miles?
   b. What is the cost if you rent the truck for 3 days and drive 75 miles?
   c. What is the cost if you rent the truck for 1 day and drive 220 miles?
   d. What is the cost if you rent the truck for 2 day and drive 160 miles?
   e. Let c be the cost of the rental and m be the number of miles driven, then the cost equation for a one day rental is $c = 95$, if the miles driven are less than 100 and $c = 95 + 0.45(m - 100)$ if the miles driven is over 100 miles.
      i. Draw the graph of the cost equation.
      ii. If you drive 210 miles what is the cost?
      iii. If the cost was $320 how far did you drive?