Homework 6

1.6 Pages 62-63:

1. \( f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \)

Reflection about X-axis

\[ \text{graph with axes and points} \]

5. \( f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \)

(Reflection about Both Axes)

\( f(\begin{bmatrix} 3 \\ 2 \end{bmatrix}) = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \)

\[ \text{graph with axes and points} \]

9. \( f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \)

Is \( x \in \mathbb{R}^2 \) such that \( f(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)?

Yes, \( w \) is in Range of \( f \)

\( x_1 + 2x_2 = 1 \)
\( x_2 = -1 \)
\( x_1 = 3 \)
16. \( f(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) \( x \in \mathbb{R}^2 \) exist?

\[
\begin{bmatrix}
0 & 2 \\
1 & 1 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 
\end{bmatrix}
\]

\( x_1 + 2x_2 = 1 \)
\( x_2 = 1 \)
\( x_1 = 0 \)
\( x_1 + x_2 = 1 \)
\( x_2 = 1 \)

but \( 0 + 2(1) = 2 \neq 1 \)

the system \( \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is inconsistent

there is no \( x \in \mathbb{R}^2 \) such that \( f(x) = w = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

So \( W \) is not in the range of \( f \).

15. a) \( A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \); \( AU = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} \)

\( AU \) is the reflection of \( U \) about the \( y \) axis.

b) \( A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \); \( AU = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \)

\( \cos \varphi = 0 \) \( \rightarrow \) \( \varphi = \frac{\pi}{2} \) \( AU \) rotates \( U \)

\( \sin \varphi = 1 \) \( \rightarrow \) \( \frac{\pi}{2} (90^\circ) \) counter-clockwise
\[
A = \begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}; \quad A^2 = \begin{bmatrix}
\cos^2 \phi - \sin^2 \phi & -2 \cos \phi \sin \phi \\
2 \cos \phi \sin \phi & \cos^2 \phi - \sin^2 \phi
\end{bmatrix}
\]

\[
T_1(u) = A^2(u) = \begin{bmatrix}
\cos(2\phi) & -\sin(2\phi) \\
\sin(2\phi) & \cos(2\phi)
\end{bmatrix} u
\]

\(A^2(u)\) rotates \(u\) counter clockwise by an angle \(2\phi\).

\(T_1(u) = A[Au]\) which rotates \(Au\) by an \(+\phi\), so \(T_1(u)\) rotates \(u\) by an \(+2\phi\).

a) \(\phi = \frac{\pi}{6} = 30^\circ\), \(T_1(u)\) rotates \(u\) counter clockwise by an angle \(\phi = \frac{\pi}{3} = 60^\circ\).

b) \(T_2(u) = A^{-1}(u)\); \(A^{-1} = \begin{bmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{bmatrix}\)

\[
A^{-1} = \begin{bmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{bmatrix} = \begin{bmatrix}
\cos(\phi) & -\sin(\phi) \\
\sin(\phi) & \cos(\phi)
\end{bmatrix}
\]

which rotate \(u\) counter clockwise by \(-\phi\), i.e., clockwise by \(\phi\).

A clockwise rotation by \(\phi\), "undoes" or inverses a counter clockwise rotation by \(\phi\).

So \(A^{-1}u\) for \(\phi = \frac{\pi}{6} = 30^\circ\), rotates \(u\) \(30^\circ\) in the clockwise direction.
19. c) For what smallest positive $k$ does $A^k(U) = U$? Answer: $k = 12$

$$A^k(U) = \begin{bmatrix} \cos(k\phi) & -\sin(k\phi) \\ \sin(k\phi) & \cos(k\phi) \end{bmatrix} (U)$$

if $k\phi = 2\pi$ then $A^k = I_2$

$$\phi = \frac{\pi}{6} \rightarrow k = \frac{2\pi}{\frac{\pi}{6}} = 12$$

20. $f(u) = Au$ for $u$ a $n \times 1 \in \mathbb{R}^n$

Prove the following: $A$ a $m \times n$ matrix

a) $f(u+v) = f(u) + f(v)$ for $u, v \in \mathbb{R}^n$

$$f(u+v) = A(u+v) = A(u) + A(v) = f(u) + f(v)$$

Theorem 1.2c

b) $f(cu) = cf(u)$ for any real number $c, u \in \mathbb{R}^n$

$$f(cu) = A(cu) = cA(u) = cf(u)$$

Theorem 1.3d

c) $f(cu + dv) = cf(u) + df(v)\quad , u, v \in \mathbb{R}^n \quad , d, c \in \mathbb{R}$

$$A(cu + dv) = A(cu) + A(dv)\quad \text{[or just apply Theorem 1.2c, (a) followed by]}$$

$$= cA(u) + dA(v) \quad \text{[Theorem 1.3d]}$$

$$= cf(u) + df(v)$$