1. (1 point)
Transform the following integrals from Cartesian coordinates to either polar (for 2D) or cylindrical (for 3D) coordinates and then evaluate them.

a) \[ \int_{-a}^{a} \int_{\sqrt{a^2-x^2}}^{0} \sqrt{a^2 - x^2 - y^2} \, dy \, dx = \] ________________

b) \[ \int_{-b}^{b} \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} \ln \left( \frac{x^2+y^2}{a^2} + 3 \right) \, dy \, dx \, dz = \] ________________

2. (1 point)
Transform the following integrals from Cartesian coordinates to spherical coordinates and then evaluate them.

a) \[ \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} z^2 \, dz \, dy \, dx = \] ________________

b) \[ \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{0}^{\sqrt{a^2-x^2-y^2}} x^2 \, dz \, dy \, dx = \] ________________
3. One can calculate the area under the Gaussian curve \( f(x) = e^{-x^2} \) by noting that
\[
\int_{0}^{\infty} f(x) \, dx = \sqrt{\int_{0}^{\infty} f(x) \, dx \int_{0}^{\infty} f(y) \, dy}
\]
then transforming the expression under the radical to polar coordinates.

(a) \( \int_{0}^{\infty} e^{-x^2} \, dx \) = ____________________________

(b) \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) = ____________________________

c) For a constant mean (average), \( \mu \), and constant standard deviation, \( \sigma \), the Normal Distribution Probability density function is given by
\[
f(x) = A \exp\left(- \frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 \right),
\]
where \( A \) is a constant chosen so that the area under the curve \( y = f(x) \) from \(-\infty \) to \( \infty \) is one. Determine the value of \( A \).

4. (3 points)
Calculate the areas (for 2 D regions) or volumes (for 3 D regions) of the following:

(a) The region in the first quadrant bounded by the lines \( y = 0 \), \( \theta = \frac{\pi}{6} \) and the cardioid \( r = 1 + \sin(\theta) \).

(b) The region in the first octant bounded by the planes \( x + y + z = 1 \) and \( 4x + 4y + z = 4 \).

(c) The spherical cap which is the intersection of \( z > b \) and the interior of a sphere of radius \( a \), \( a > b \), centered at the origin.
d) The intersection of the region bounded by a sphere of radius $a$ centered at the origin and a right circular cylinder of radius $b$, $b < a$, whose axis is the $z$ axis.

e) The region in the first octant bounded by the plane $z = 0$, the cylinder $r = 3$, and the plane which intersects the coordinate axes at $(0, 0, 6)$, $(6, 0, 0)$ and $(0, 6, 0)$.

f) The interior of the sphere, $\rho \leq a$, that lies between the two cones $\phi = \frac{\pi}{4}$ and $\phi = \frac{\pi}{3}$.

5. (2 points)
Let $D$ be the region in the first quadrant bounded by the $x$ axis, $y = x$ and $y = \frac{4}{x}$.

a) Evaluate $\int \int_D y \, dA = \phantom{a\ b\ c\ d}$

b) Given $u = xy$ and $v = x$, determine the Jacobean for the transformation from $(x, y)$ to $(u, v)$.

c) Using the substitution of part b) sketch the region of the $uv$ plane which is the image of $D$.

d) Calculate the value of $\int \int_D y \, dA$ in the transformed coordinates $u, v$. 
6. (3 points)
Let $D$ be the region $1 \leq x - y \leq 2$ and $1 \leq x + y \leq 3$.

a) Evaluate $\int \int_D (x^2 - y^2) \, dA = \underline{\phantom{\underline{\phantom{\ldots}}}}$

b) Given $u = x + y$ and $v = x - y$, determine the Jacobian for the transformation from $(x, y)$ to $(u, v)$.

c) Using the substitution of part b) what region of the $uv$ plane is the image of $D$?

d) Calculate the value of $\int \int_D (x^2 - y^2) \, dA$ in the transformed coordinates $u, v$.

7. (1 point)
Using the transformed variables, $u = x/a$, $v = y/b$, $w = z/c$, evaluate $\int \int \int |xyz| \, dV$ over the solid ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

8. (5 points)
a) For $a > 0$, calculate the centroid of the region in the first quadrant bounded by the coordinate axes and $y = e^{-ax}$.
b) Calculate the centroid of a uniform density right circular cone of base radius $a$ and height $h$, if the cone axis is the $z$ axis and the base of the cone rests in the $xy$ plane.

c) Calculate the radius of gyration of a uniform density right circular cone of base radius $a$ and height $h$, if the axis of rotation is the cone axis.

d) Calculate the radius of gyration of a uniform density solid sphere of radius $a$ rotated about an axis through its center.

e) Calculate the radius of gyration of a uniform density solid sphere of radius $a$ rotated about an axis tangent to the sphere.
9. (3 points)
a) Calculate the centroid of a uniform density slab of material in the shape of an equilateral triangle of side $S$. Place one side of the triangle along the $x$ axis so that the top vertex is in the first quadrant and the origin is at the left vertex.

b) Calculate the radius of gyration of the equilateral triangle about an axis perpendicular to the plane of the triangle and passing through the origin.

c) Calculate the radius of gyration of the equilateral triangle about an axis perpendicular to the plane of the triangle and passing through the center of mass.

10. (3 points) Consider a uniform density right circular cylinder of base radius $a$ and height $h$. 

a) Calculate the radius of gyration of the cylinder about an axis perpendicular to the axis of the cylinder and passing through the center of mass.

b) Evaluate the limit of your answer to part a) as $a$ tends to zero.

c) Calculate the radius of gyration of the cylinder about an axis perpendicular to the axis of the cylinder and attached to the end of the cylinder in such a way that it passes through the center of the circle of radius $a$. 

11. (1 point)
Integrate the scalar function $e^{y^2 + z^2 - 4x}$ over that part of the surface of the cylinder $y^2 + z^2 = 4$ in the first octant.

12. (2 points)
For positive $a$ and $h$, let $A$ designate the region of $\mathbb{R}^3$ enclosed by the elliptic hyperboloid, $x^2 + y^2 - z^2 = a^2$, and the two planes, $z = -\frac{h}{2}$ and $z = \frac{h}{2}$. Let $B$ represent the orientable surface of $A$.

a) Determine the volume of $A$.

b) For the position vector field $\vec{F} = \vec{R} = x\hat{i} + y\hat{j} + z\hat{k} = \langle x, y, z \rangle$, calculate the flux out of the lateral surface of $A$.

c) For the position vector field $\vec{F} = \vec{R}$, calculate the flux out of the top surface of $A$.

d) For the position vector field $\vec{F} = \vec{R}$, calculate the flux out of the bottom surface of $A$.

e) What is the total outward flux of $\vec{F} = \vec{R}$ over $B$?

$$\int \int_B \vec{F} \cdot \vec{n} \, dS$$

f) What is the ratio of your answer to part e) and your answer to part a)?
13. (1 point) Integrate the scalar function \( f(x, y, z) = x \) along the path \( \vec{R} = 5t \hat{i} + t^2 \hat{j} - t \hat{k} \) from \((0, 0, 0)\) to \((15, 9, -3)\).

14. (1 point)
Calculate the flow integral \( \int \vec{F} \cdot \vec{T} \, ds \) for the vector field \( \vec{F} \) along the path specified. (\( \vec{F} \cdot \vec{T} \, ds = \vec{F} \cdot d\vec{r} = F_x \, dx + F_y \, dy \))

a) \( \vec{F} = 6x \hat{i} - 9y \hat{j} \) the fourth quadrant path from \((5, -3)\) to \((8, 0)\) along the curve \( x^2 - 10x + y^2 + 16 = 0 \)

b) \( \vec{F} = x \sin(x) \hat{i} - x \cos(y) \hat{j} + \sqrt{xy} \, z \hat{k} \) from \((0, 0, 0)\) to \((\pi, \pi, \pi)\) along \( y = x \) and \( z = \frac{x^2 + y^2}{2\pi} \).

15. (5 points)
For each closed curve and vector field stated calculate both the counterclockwise circulation \( \oint \vec{F} \cdot \vec{T} \, ds \) and the outward flux \( \oint \vec{F} \cdot \vec{n} \, ds \).

a) \( C \) : The circle of radius \( a \) centered at \((h, k)\).
\( \vec{F} = x \hat{i} - y \hat{j} \)
\( \oint \vec{F} \cdot \vec{T} \, ds = \)
\( \oint \vec{F} \cdot \vec{n} \, ds = \)
b) $C: \text{ The circle of radius } a \text{ centered at } (h, k)$.
\[
\vec{F} = x\hat{i} + y\hat{j}
\]
\[
\oint_C \vec{F} \cdot \hat{T} \, ds =
\]
\[
\oint_C \vec{F} \cdot \hat{n} \, ds =
\]

c) $C': \text{ The circle of radius } a \text{ centered at } (h, k)$.
\[
\vec{F} = y\hat{i} - x\hat{j}
\]
\[
\oint_{C'} \vec{F} \cdot \hat{T} \, ds =
\]
\[
\oint_{C'} \vec{F} \cdot \hat{n} \, ds =
\]

d) $C: \text{ The circle of radius } a \text{ centered at } (h, k)$.
\[
\vec{F} = y\hat{i} + x\hat{j}
\]
\[
\oint_C \vec{F} \cdot \hat{T} \, ds =
\]
\[
\oint_C \vec{F} \cdot \hat{n} \, ds =
\]

e) $C: \text{ The ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2y}{b}$.
\[
\vec{F} = xy\hat{i} + 2x^2\hat{j}
\]
\[
\oint_C \vec{F} \cdot \hat{T} \, ds =
\]
\[
\oint_C \vec{F} \cdot \hat{n} \, ds =
\]
16. (5 points) For each vector field indicate which have a non-zero circulation and which have a non-zero flux.

\( \vec{F}(x,y) = (y, x) = -3y \hat{i} + 3x \hat{j} \)

a)

\( \vec{F}(x,y) = \left( \frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right) = \frac{y}{\sqrt{x^2 + y^2}} \hat{i} + \frac{x}{\sqrt{x^2 + y^2}} \hat{j} \)

b)

\( \vec{F}(x,y) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) = \frac{x}{\sqrt{x^2 + y^2}} \hat{i} + \frac{y}{\sqrt{x^2 + y^2}} \hat{j} \)

c)

d)
e) \( \vec{F}(x, y) = \left( y - \frac{x}{2}, -x - \frac{y}{2} \right) = \left( y - \frac{x}{2} \right) \hat{i} + \left( -x - \frac{y}{2} \right) \hat{j} \)

f) \( \vec{F}(x, y) = \left( \cos(y), \sin(x) \right) = \cos(x) \hat{i} + \sin(y) \hat{j} \)

g) \( \vec{F}(x, y) = \left( \cos(y), \sin(x) \right) = \cos(y) \hat{i} + \sin(x) \hat{j} \)

h) \( \vec{F}(x, y) = \left( \frac{y}{\sqrt{\sin^2(x) + y^2}}, \frac{-\sin(x)}{\sqrt{\sin^2(x) + y^2}} \right) - \frac{y}{\sqrt{\sin^2(x) + y^2}} \hat{i} - \frac{\sin(x)}{\sqrt{\sin^2(x) + y^2}} \hat{j} \)
i) \[ \vec{F}(x,y,z) = (x,y,z) = x\hat{i} + y\hat{j} + z\hat{k} \]

j) \[ \vec{F}(x,y,z) = (-y,x,z) = -y\hat{i} + x\hat{j} + z\hat{k} \]
17. (2 points) Consider the vector field: \( \vec{F} = (z - y) \hat{k} + (x - z) \hat{j} + (y - x) \hat{i} = < z - y, x - z, y - x >. \)

a) Calculate the magnitude of \( \nabla \times \vec{F} \)

b) Calculate the direction of \( \nabla \times \vec{F} \)

Go to the website, [http://www.monroecc.edu/faculty/paulseeburger/calcnsf/CalcPlot3D/](http://www.monroecc.edu/faculty/paulseeburger/calcnsf/CalcPlot3D/), and clear the default surface graph displayed by clicking on the "clear the plot" icon. From the Add to graph menu select Vector Field.

Use the Magnifier to Zoom out once to form a 3-D box from -4 to 4 in every direction.

Enter \( z - y \) for M = x component, enter \( x - z \) for N = y component, enter \( y - x \) for P = z component.

Set the number of vectors to 7 along each axis. Click on the box to the left of \( M, N, P \) to generate the vector field.

Next from the Add to graph menu select Space Curve: \( r(t) \). Enter \( t \) for each coordinate and let \( t \) run from -4 to 4.

Click on the box to the left of \( x(t) = t \) to generate the line \( x = y = z \) from on corner of the viewing box to the opposite side.

c) Adjust the perspective with the mouse so that you are looking down the line \( x = y = z \). Explain how what you see is consistent with your calculations in parts a) and b).