1. (2 points)
   a) Calculate the surface integral $\int_S \int (x\hat{i} + y\hat{j}) \cdot \hat{n} \, dS$ over the surface of the cylinder $x^2 + y^2 = a^2$ with $0 \leq z \leq h$.

   b) Does this answer make sense geometrically? Explain.

   c) Calculate the surface integral $\int_S \int (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{n} \, dS$ over the surface of the sphere of radius $a$ centered at the origin.

   d) Does this answer make sense geometrically? Explain.

2. (2 points)
   Let $\Phi(x, y) = \frac{\ln(x^2 + y^2)}{2}$,
   a) Calculate the two dimensional gradient of $\Phi$, $\nabla \Phi = \ldots$

   b) Calculate the 2-D Laplacian of $\Phi$, $\nabla^2 \Phi = \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \ldots$

   c) Let $C$ be the circle of radius $a$ centered at the origin. Calculate the following counterclockwise flux integral across $C$, $\oint_C \nabla \Phi \cdot \hat{n} \, ds = \ldots$

   d) For the interior, $D$, of the circle $C$ calculate the integral $\int \int_D \nabla \cdot \nabla \Phi \, dA = \ldots$

   e) Are the results of parts c) and d) inconsistent with the Divergence (Gauss's) Theorem? Explain.

   f) Consider any simple closed curve $Q$ which encloses $C$. What is the counterclockwise flux integral across $Q$?
   $\oint_Q \nabla \Phi \cdot \hat{n} \, ds = \ldots$
3. (2 points)
Let \( \Phi(x, y, z) = \frac{-1}{\sqrt{x^2+y^2+z^2}} \).

a) Calculate the three dimensional gradient of \( \Phi \),
\[
\nabla \Phi = \ldots
\]

b) Calculate the 3-D Laplacian of \( \Phi \),
\[
\nabla^2 \Phi = \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \ldots
\]

c) Let \( C \) be the sphere of radius \( a \) centered at the origin. Calculate the flux integral across the surface of \( C \),
\[
\iint_C \nabla \Phi \cdot \mathbf{n} \, dS = \ldots
\]

d) For the interior, \( D \), of the sphere \( C \) calculate
\[
\iiint_D \nabla \cdot \Phi \, dV = \ldots
\]

e) Are the results of parts c) and d) inconsistent with the Divergence (Gauss’s) Theorem? Explain.

f) Consider any simple closed surface \( Q \) which encloses \( C \). What is the flux integral across \( Q \)?
\[
\iint_Q \nabla \Phi \cdot \mathbf{n} \, dS = \ldots
\]

4. (2 points)
The magnetic field about an infinite line of current along the \( z \) axis varies like \( \vec{B}(x, y) = -\frac{y\hat{x} + x\hat{y}}{x^2+y^2} \).

a) Calculate the two dimensional curl of \( \vec{B} \),
\[
\nabla \times \vec{B} = \ldots
\]

b) Let \( C \) be the circle of radius \( a \) centered at the origin in the \( xy \) plane. Calculate the counterclockwise circulation around \( C \)
\[
\oint_C \vec{B} \cdot \mathbf{T} \, ds = \ldots
\]

c) For the interior, \( D \), of the circle \( C \) calculate
\[
\iint_D (\nabla \times \vec{B}) \cdot \mathbf{k} \, dS = \ldots
\]

d) Are the results of parts b) and c) inconsistent with Stokes’ Theorem? Explain.

f) Consider any simple closed curve \( Q \) in the \( xy \) plane which encloses \( C \). What is the counterclockwise circulation around \( Q \)?
\[
\oint_Q \vec{B} \cdot \mathbf{T} \, ds = \ldots
\]
5. (1 point)
Suppose $\vec{F}$ is a radial vector field in the $xy$ plane, i.e. $\vec{F} = f(x^2 + y^2)(\hat{x} + y\hat{j})$ for some differentiable scalar function $f$.

a) Explain why such a field is called radial.

b) Is $\vec{F}$ a conservative field? Explain.

6. (2 points)
Consider the force field $\vec{F} = b(\frac{y}{z}\hat{i} + \frac{x}{z}\hat{j} - \frac{xy}{z^2}\hat{k}) = \langle \frac{by}{z}, \frac{bx}{z}, -\frac{bxy}{z^2} \rangle$ for some constant $b$.

a) Calculate the work done by this field in going from $\vec{r}(0) = (1, 2, 3)$ to $\vec{r}(1) = (4, 9, 6)$ along the path $\vec{r} = (-1 + 5t)\hat{i} + (2 + 7t)\hat{j} + (3 + 3t)\hat{k}$ for $0 \leq t \leq 1$.

b) Calculate the work done by this field in going from $\vec{r}(0) = (1, 2, 3)$ to $\vec{r}(1) = (4, 9, 6)$ along a path made up of three line segments each parallel to the coordinate axes, starting with the path parallel to the $x$ axis and ending with the path parallel to the $z$ axis.

c) Explain the relationship between the answers to a) and b).

d) Calculate the work done by this field in going from $\vec{r}(0) = (1, 2, 3)$ to $\vec{r}(1) = (4, 9, 6)$ along any path.

7. (1 point)
For constant positive $a$, consider the force field $\vec{F} = (x^2y + \frac{b^3}{3})\hat{i} + ax^2\hat{j}$.

a) Is this a conservative force? Explain.

b) Find the simple closed path in the $xy$ plane along which the work done by $\vec{F}$ in a counterclockwise circulation is a maximum.
8. (2 points)
Let $H$ be that portion of the sphere with boundary $x^2 + y^2 + z^2 = a^2$ and $\frac{a}{2} \leq z \leq a$. Let $B$ represent the oriented surface of $H$ and $C$ be the intersection of $z = \frac{a}{2}$ and the surface of the sphere $\rho = a$.
For $\vec{F} = xz\hat{i} - xz\hat{j} + yz\hat{k} = <xz, -xz, yz>$ calculate the following:

a) \[ \int \int_B \vec{F} \cdot \hat{n} \, dS \]

b) \[ \oint_C \vec{F} \cdot \hat{T} \, ds \]
9. (3 points)
Show by explicit calculation that the following 3D vector identities are valid. Φ and Ψ represent 3D scalar functions and \( \vec{A} \) and \( \vec{B} \) represent 3D fields.

a) \( \nabla \cdot \nabla \Phi = \nabla^2 \Phi \)

b) \( \nabla \times \nabla \Phi = 0 \)

c) \( \nabla \cdot (\nabla \times \vec{A}) = 0 \)

d) \( \nabla (\Phi \Psi) = \Phi \nabla \Psi + \Psi \nabla \Phi \)

e) \( \nabla \cdot \left( \vec{A} \times \vec{B} \right) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B} \)

f) \( \nabla \cdot (\Phi \vec{A}) = \Phi \nabla \cdot \vec{A} + \nabla \Phi \cdot \vec{A} \)

g) \( \nabla \times (\Phi \vec{A}) = \Phi \nabla \times \vec{A} + \nabla \Phi \times \vec{A} \)

h) \( \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \)

Here, \( \nabla^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2} \).
10. (3 points) Maxwell's Equations (in free space) for the electromagnetic fields in rationalized MKS units are given by the following:

\[
\nabla \cdot \vec{E} = \frac{\rho_e}{\epsilon_0} \quad ; \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ; \quad \nabla \cdot \vec{B} = 0 \quad ; \quad \nabla \times \vec{B} = \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)
\]

Here \( \vec{E} \) is the electric field, \( \vec{B} \) is the magnetic field, \( t \) is time, \( \rho_e \) is the electric charge density, \( \vec{J} \) is the electric current density, \( \epsilon_0 \) is the permittivity of free space \( \left( 8.854 \times 10^{-12} \text{ F/m} \right) \), \( \mu_0 \) is the permeability of free space \( \left( 4\pi \times 10^{-7} \text{ H/m} \right) \). The amount of electric charge \( Q \) in a region of space \( R \) is given by the volume integral \( \int \int \int \rho_e \, dV \), while the total electric current (charge per unit of time) flowing through a surface \( S \) is given by the surface integral \( \int \int \vec{J} \cdot \vec{n} \, d\sigma \).

a) Consider a charge density which is spherically symmetric about the origin and which vanishes for \( \rho (\text{the radial distance from the origin}) > a \). Let \( Q = 4\pi \int_0^a \rho_e(\rho) \rho^2 \, d\rho \). What is the electric field at \( \vec{R} \) where \( |\vec{R}| > a \)?

b) Show that electric charge conservation is a consequence of Maxwell's equations. That is, show that the rate at which electric charge changes in a region of space is equal to the rate at which charge is flowing into or out of the region.

c) Under what conditions can the electric field be written as \( -\nabla \Phi \) where \( \Phi \) is a scalar potential function?

d) The EMF (electro motive force) measured in volts is the line integral \( \oint_C \vec{E} \cdot \vec{T} \, ds \) around the closed path \( C \). How is this EMF related to what the magnetic field is doing?
Solve the following initial value problems. Each problem is worth 1 point.

11. \[ \frac{df}{dx} = |x| \quad \text{with} \quad f(-3) = 2. \]

12. \[ \frac{dy}{dx} = \frac{x-y}{x+y} \quad \text{with} \quad y = -4 \text{ at } x = 0. \]

13. \[ xy' + 2y = x^2 \quad \text{with} \quad y(-2) = 1. \]

14. \[ \frac{df}{dx} = e^{-2x} - 2f(x) \quad \text{with} \quad f(0) = -1. \]

15. \[ \frac{dy}{dx} = -2y + \sin(x) \quad \text{with} \quad y(0) = \frac{4}{5}. \]
16. \( y' + y = |x - 3| \) with \( y(0) = 4 \).

17. \( \frac{dy}{dx} + y = 4e^{-x} \) with \( y(0) = \alpha \).

18. (2 points)
   a) Solve \( \frac{dy}{dx} + 2xy = x \) subject to \( y(0) = a \).

   b) Generate the direction field graph for this equation with a window from \(-2.5\) to \(2.5\) in both the \(x\) and \(y\) directions. On this same graph plot the five functions \( y(x) \) for \( a = -0.5, 0, 0.5, 1 \) and \( a = 1.5 \).
   In WinPlot use the 2-dim Window and from the Equa menu select \(\text{Differential } dy/dt\). In the differential equation menu, set \(x'\) equal to 1 and \(y'\) equal to \(\frac{dy}{dx}\). Select the "vectors" check box. Change the window using the command sequence \(\textbf{View}, \text{Set corners}\). Initial values can be specified through the \(\text{One/dy/dt trajectory}\) menu. Just set \(x\) equal to zero (for this example) and \(y\) equal to \(a\) (ignore the setting for \(t\)), then press the "draw" button with "both" checked. Repeat this procedure for each initial condition. Choosing the IVPs a different color from the slope field makes the display easier to interpret.

   c) Explain what happens at \( a = 0.5 \).

19. (3 points)
   a) Given \( \frac{dy}{dx} + 4y = 8x + 6 \) with \( y(0) = a > 0 \), let \( u(x) \) be the linear function that \( y(x) \) approaches as \( x \to \infty \). For what value of \( a \) is \( y = u(x) \)?
b) Given \( \frac{dy}{dx} + y = 6 \), determine \( \lim_{x \to \infty} y(x) \) and \( \lim_{x \to \infty} \frac{dy}{dx} \).

c) Find the value of \( a \) that makes \((ax^2 + 4xy)dx + (ax^2 + y^2)dy = 0\) an exact first order equation. For this value of \( a \) solve for the implicit curve \( f(x, y) = 0 \) that passes through the point \((1, 2)\).

20. (1 point)
For each family of curves \( f(x, y) = c \) determine the equation for the family of orthogonal trajectories. Sketch several curves from each family.

a) \( f(x, y) = x^2 - y^2 = c \)

b) \( f(x, y) = e^x y = c \)
21. (1 point) Linear Circuits:

a) A resistor $R$ (10.0 $\Omega$) and an inductor $L$ (0.250 H) are hooked up in series to a power supply $V$ (20.0 volts) which establishes a steady state current of $\frac{V}{R}$ (2.0 amperes). The power supply is then removed and the current decays in a manner described by:

$$Ri + L \frac{di}{dt} = 0.$$ How long after the power supply is removed is the current down to 0.50 amperes? Note: $\frac{1}{1 \Omega} = 1$ second. For arbitrary values of $R$, $L$ and $V$, how long after the power supply is removed is the current down to $\frac{V}{4R}$?

b) A capacitor $C$ (8.0 $\mu$F) is charged to a potential difference $V$ (12.0 volts). The power supply is removed and the capacitor is allowed to discharge through a resistor $R$ (1.0 M$\Omega$). The charge $Q$ on the capacitor is given by $Q = CV$ and decays in a manner described by:

$$R \frac{dQ}{dt} + \frac{Q}{C} = 0.$$ How long after the power supply is removed is the charge on the capacitor down to 0.250 its starting value? Note: $1 \text{F} \cdot 1 \Omega = 1$ second. For arbitrary values of $R$, $C$ and $V$ (the power supply voltage), how long after the power supply is removed is the charge down to $\frac{CV}{4}$?

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22. (2 points) Population Growth

a) The most naive (and most frightening) model for population growth is the exponential growth model. This assumes that the rate of growth is directly proportional to the current population size. In symbols this is expressed as $\frac{dN}{dt} = kN$, where $N$ is the number of individuals in the population at time $t$ and $k$ is a constant. Roughly, $k$ is the number of new ‘offspring’ produced by each individual in a relevant unit of time. Solve $\frac{dN}{dt} = kN$, subject to the initial condition that $N(0) = N_0$.

b) The exponential model is obviously unrealistic for large times in that it leads to unbounded population growth. There is only so much matter in the universe! At some point limited food and living space limit the population size of any species. One way to model this behavior mathematically is to modify the rate equation as follows: $\frac{dN}{dt} = kN \left(1 - \frac{N}{L}\right)$, where $L$ is the limiting population size. Initially $N$ increases exponentially as before; however, as $N$ approaches $L$ the rate of growth slows to zero. Hence, $N = L$ is a horizontal asymptote of the solution. This rate equation is sometimes called the logistic equation. For the logistic equation model what value of $N$ makes the rate of growth a maximum?

What is this maximum rate of growth? How does this compare to the rate of growth of the exponential model for the same population size?
Solve the logistic equation, subject to the initial condition that \( N(0) = N_0 \).

Evaluate \( \lim_{t \to \infty} N(t) \) = ________________

For \( N_0 = 500 \), \( L = 500,000 \) and \( k = 0.15 \text{ year}^{-1} \), according to the exponential model how long would it take the population to double to 1000? To increase to 300,000?

For \( N_0 = 500 \), \( L = 500,000 \) and \( k = 0.15 \text{ year}^{-1} \), according to the logistic model how long would it take the population to double to 1000? To increase to 300,000?

For \( N_0 = 500 \), \( L = 500,000 \) and \( k = 0.15 \text{ year}^{-1} \), make a careful graph of the solutions of both the exponential and logistic models. A graphing calculator or computer program would be helpful here.
23. (2 points)
Perform the following operations on the given complex numbers and express the result in standard rectangular form $(a + bi)$.

a) $(6 + 7i) - (-3 - \sqrt[4]{-64}) = \underline{\hspace{6cm}}$

b) $(3 + 2i)(2 + \sqrt{-9}) = \underline{\hspace{6cm}}$

c) $\frac{2 - 3i}{\sqrt{16 - \sqrt{-25}}} = \underline{\hspace{6cm}}$

d) $(3 - 7i)^2 = \underline{\hspace{6cm}}$

e) $(2i)^5 = \underline{\hspace{6cm}}$

f) $i^{-1921} = \underline{\hspace{6cm}}$

g) $[\cos(\frac{\pi}{3}) + isin(\frac{\pi}{3})]^2 = \underline{\hspace{6cm}}$

h) $\sin(2i) = \underline{\hspace{6cm}}$

24. (2 points)
For each of the following complex numbers
i) Express the result in standard rectangular form
ii) Express the result in standard exponential form $(re^{i\theta})$
iii) Graph the number

a) $(2 + i)(3 + i) = \underline{\hspace{6cm}}$(Rectangular) = \underline{\hspace{6cm}}(Exponential)
25. (1 point)
For $x$ real express
a) $\sin(ix)$ in terms of $\sinh(x)$.

b) $\cos(ix)$ in terms of $\cosh(x)$.

c) sinh$(ix)$ in terms of $\sin(x)$.

d) cosh$(ix)$ in terms of $\cos(x)$.

Solve the following initial value problems. Each problem is worth 1 point.
26. $y'' - y' - 6y = 0$  with  $y(0) = 5$, $y'(0) = -5$.

27. $\frac{d^2 f}{dt^2} + 4 \frac{df}{dt} + 13 f = 40\cos(3t)$  with  $f(0) = 2$, $\left. \frac{df}{dt} \right|_0 = -1$.
28. \( y'' + 4y' + 4y = 32e^{2x} \) with \( y(0) = 4, \ y'(0) = -3 \).

29. \( y'' - 3y' - 4y = 10e^{4x} \) with \( y(0) = 2, \ y'(0) = 6 \).

30. \( y'' + 8y' + 16y = 2e^{-4x}\sin(x) \) with \( y(0) = 3, \ y'(0) = -12 \).

31. \( y'' + 4y = 16x\sin(2x) \) with \( y(0) = -2, \ y'(0) = 4 \).

32. \( y'' - 2y' + y = 48x^2e^x \) with \( y(0) = 2, \ y'(0) = 3 \).
33. (3 points) **Transient LRC Circuit.**

A capacitor, $C$, an inductor, $L$, and a resistor, $R$, are hooked up in series. The capacitor is charged to a DC voltage of $V_0$ and then allowed to discharge through the resistor and inductor. The current, $I$, satisfies the second order, linear homogeneous ODE:

$$L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = 0,$$

with $I(0) = 0$ and $I'(0) = \frac{V_0}{L}$.

If $R = 0$, one has an LC circuit in which the current varies sinusoidally at the resonance frequency $\omega_0 = \frac{1}{\sqrt{LC}}$. If $R \neq 0$, $R$ cannot be negative) the current decays exponentially in time.

a) What is the condition on $R$ in terms of the values of $L$ and $C$ so that the current is underdamped?

b) What is the current as a function of time when the circuit is underdamped?

c) What is the current as a function of time when the circuit is critically damped?

d) What is the current as a function of time when the circuit is overdamped?

e) If $L = 4.8\, \text{mA}H$ and $C = 1.2\, \mu\text{F}$ calculate the value of the resistance for critical damping. $R = \underline{\hspace{2cm}}$

f) For $L = 4.8\, \text{mA}H$, $C = 1.2\, \mu\text{F}$, $V_0 = 10.0$ volts and each value of $R$ in the table below, calculate the missing parameters in the units requested. Here $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$. If $\omega = 1456.89\, \text{sec}^{-1} = \frac{1456.89}{1\, \text{sec}} = \frac{1456.89 \times 10^{-3}}{1\, \text{sec} \times 10^{-3}} = 1.45689\, \text{msec}^{-1}$.

<table>
<thead>
<tr>
<th>$R$ (\Omega)</th>
<th>$\omega$ (\text{sec}^{-1})</th>
<th>$\frac{R}{2L}$ (\text{sec}^{-1})</th>
<th>$\frac{R}{2L}$ (\text{msec}^{-1})</th>
<th>$\frac{V_0}{\sqrt{\frac{L}{C} - \frac{R^2}{4L^2}}}$ (amps)</th>
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g) Generate on 1 plot six graphs of current versus time for $L = 3.4\, \text{mA}H$, $C = 1.2\, \mu\text{F}$, $V_0 = 10.0$ volts and each value of $R$ in the table above. For all graphs plot the current from 0 msec to 1.6 msec.
34. (3 points) **Driven LRC Circuit**

A capacitor, $C$, an inductor, $L$, and a resistor, $R$, are hooked up in series with an AC power supply. The current, $I$, satisfies the second order, linear ODE:

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \frac{dV}{dt},$$

where $V$ is the AC voltage function. Suppose $V = V_0 \cos(\omega t)$ for an angular velocity $\omega$ and an amplitude $V_0$ fixed by the power supply. Replace $V$ by the complex voltage $V_0 e^{i\omega t}$ and solve the resulting complex ODE by assuming a particular complex solution of the form $A e^{i\omega t}$. Solve for $A$ in terms of $L$, $C$, $R$, $V_0$, and $\omega$. The particular solution for the actual current is then given by $I(t) = \text{Real Part of } (A e^{i\omega t})$.

b) What is the amplitude of the driven current?

c) Relative to the power supply voltage what is the phase shift of the current?

d) What value of $\omega$ gives the most current for all other circuit constants held fixed? This is called the resonant frequency of the circuit.

e) Explain why in this analysis we ignored the homogeneous solution.