Directional Derivatives as Vectors

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Statement of problem

- We are given a surface defined by an equation of the form \( z = f(x, y) \)
- The definition of *differentiable* or Total Derivative is a statement about how \( \Delta z \) can be written:
  \[
  \Delta z = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \text{where} \quad \varepsilon_1 \to 0 \text{ as } \Delta x \to 0, \\
  \Delta y \to 0 \quad \text{and} \quad \varepsilon_2 \to 0 \text{ as } \Delta x \to 0, \Delta y \to 0
  \]
- Not a very satisfying definition
- We’ve already interpreted \( f_x \) and \( f_y \) as “slopes” of tangent vectors to the surface for certain level curves/“directions”
- Goal: find tangent vectors to the surface in any “direction”
- Result: a more satisfying definition of *differentiable*
Surface, level curves, tangent vectors

- $z = f(x, y)$
- We assume that $f$ is \textit{differentiable} or has Total Derivative
- $f_x$ and $f_y$ denote the partial derivatives of $f$ with respect to $x$ and $y$, respectively
- $(a, b)$ is a fixed point in the domain of $f$
- Level curve for fixed $a$ is $L_{1,a}$
  - $L_{1,a}(y) = f(a, y)$
  - Has a tangent vector at $(a, b, f(a, b))$ of $\langle 0, 1, f_y(a, b) \rangle$
- Level curve for fixed $b$ is $L_{2,b}$
  - $L_{2,b}(x) = f(x, b)$
  - Has a tangent vector at $(a, b, f(a, b))$ of $\langle 1, 0, f_x(a, b) \rangle$
The directional derivative

- \( z = f(x, y) \)
- Unit vector in the xy-plane having direction \( \theta \) is \( u = \langle \cos \theta, \sin \theta \rangle \)
- The gradient of \( f \) is denoted by \( \nabla f \)
  - \( \nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \)
- The directional derivative of \( f \) in the direction of \( u \), \( D_u(f)(x, y) \), is \( \nabla f(x, y) \cdot u \), the dot product of two vectors
  - \( D_u(f)(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta \)
- Remember: \( D_u(f)(x, y) \) is a number giving the rate of change of \( f(x, y) \) in the direction of \( u \). It is not a vector!
A specific surface

- \( z = f(x, y) = \frac{x^2}{4} + \frac{y^2}{9} \)
- \( f_x = \frac{x}{2} \)
- \( f_y = \frac{2y}{9} \)
- \((a, b, f(a, b))\) is a fixed point on the surface
- Level curve for fixed \(a\) is \(L_{1,a}\)
  - \(L_{1,a}(y) = \frac{a^2}{4} + \frac{y^2}{9}\)
  - Tangent vector at \((a, b, f(a, b))\) is \(\langle 0, 1, \frac{2b}{9} \rangle\)
- Level curve for fixed \(b\) is \(L_{2,b}\)
  - \(L_{2,b}(x) = \frac{x^2}{4} + \frac{b^2}{9}\)
  - Tangent vector at \((a, b, f(a, b))\) is \(\langle 1, 0, \frac{a}{2} \rangle\)
The directional derivative (specific)

- \( z = f(x, y) = \frac{x^2}{4} + \frac{y^2}{9} \)
- Unit vector in the \( xy \)-plane having direction \( \theta \) is
  \[ u = (\cos \theta, \sin \theta) \]
- \( \nabla f(x, y) = \left( \frac{x}{2}, \frac{2y}{9} \right) \)
- \( D_u(f)(x, y) = \frac{x}{2} \cos \theta + \frac{2y}{9} \sin \theta \)
- Remember: \( D_u(f)(x, y) \) is a number, not a vector!
Two special directional derivatives

- \( z = f(x, y) \)
  - When \( \theta = 0 \) we have \( D_u(f)(x, y) = f_x(x, y) \)
  - When \( \theta = \frac{\pi}{2} \) we have \( D_u(f)(x, y) = f_y(x, y) \)

- \( z = f(x, y) = \frac{x^2}{4} + \frac{y^2}{9} \)
  - When \( \theta = 0 \) we have \( D_u(f)(x, y) = \frac{x}{2} \)
  - When \( \theta = \frac{\pi}{2} \) we have \( D_u(f)(x, y) = \frac{2y}{9} \)
Two special tangent vectors

- Recall from slide 3
  - \( z = f(x, y) \)
  - \((a, b)\) is a fixed point in the domain of \( f \)
  - \( L_{1,a} \) and \( L_{2,b} \) are level curves
  - A tangent vector to \( L_{1,a} \) at \((a, b)\) is \( \langle 0, 1, f_y(a, b) \rangle \)
  - A tangent vector to \( L_{2,b} \) at \((a, b)\) is \( \langle 1, 0, f_x(a, b) \rangle \)
- \( L_{2,b} \) corresponds to direction \( \theta = 0 \)
- \( L_{1,a} \) corresponds to direction \( \theta = \frac{\pi}{2} \)
Two special tangent vectors (continued)

- \( L_{1,a}(y) = f(a, y) \) and \( \theta = \frac{\pi}{2} \)
- \( R_{1,a}(t) = \langle a, t, f(a, t) \rangle \) is the vector function for \( L_{1,a} \)
- \( R'_{1,a}(t) = \langle 0, 1, f_y(a, t) \rangle \)
- Direction vector \( u = \langle \cos \frac{\pi}{2}, \sin \frac{\pi}{2} \rangle = \langle 0, 1 \rangle \)
- Tangent vector to \( L_{1,a} \) at \( (a, b, f(a, b)) \) is \( R'_{1,a}(b) = \langle 0, 1, f_y(a, b) \rangle \)
- \( R'_{1,a}(b) \) is also a tangent vector to the surface at the point \( (a, b, f(a, b)) \)
- \( R'_{1,a}(b) = \langle 0, 1, f_y(a, b) \rangle = \langle \cos \frac{\pi}{2}, \sin \frac{\pi}{2}, D_u(f)(a, b) \rangle \)
Two special tangent vectors (continued)

- \( L_{2,b}(x) = f(x, b) \) and \( \theta = 0 \)
- \( R_{2,b}(t) = \langle t, b, f(t, b) \rangle \) is the vector function for \( L_{2,b} \)
- \( R'_{2,b}(t) = \langle 1, 0, f_x(t, b) \rangle \)
- Direction vector \( \mathbf{u} = \langle \cos 0, \sin 0 \rangle = \langle 1, 0 \rangle \)
- Tangent vector to \( L_{2,b} \) at \((a, b, f(a, b))\) is \( R'_{2,b}(b) = \langle 1, 0, f_x(a, b) \rangle \)
- \( R'_{2,b}(b) \) is also a tangent vector to the surface at the point \((a, b, f(a, b))\)
- \( R'_{2,b}(a) = \langle 1, 0, f_x(a, b) \rangle = \langle \cos 0, \sin 0, D_u(f)(a, b) \rangle \)
The general tangent vector

- \( z = f(x, y) \) and general \( \theta \)
- \( \mathbf{u} = \langle \cos \theta, \sin \theta \rangle \)
- \( D_{\mathbf{u}}(f)(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta \)
- \( F(x, y, z) = f(x, y) - z \)
- \( \nabla F(x, y, z) = \langle f_x(x, y), f_y(x, y), -1 \rangle \)
- The plane tangent to the surface at the point \( (a, b, f(a, b)) \) has a normal vector
  \( \nabla F(a, b, f(a, b)) = \langle f_x(a, b), f_y(a, b), -1 \rangle \)
- The vertical plane containing \( (a, b, f(a, b)) \) and parallel to \( \mathbf{u} \) has a normal vector \( \langle -\sin \theta, \cos \theta, 0 \rangle \)
The vertical plane intersects the surface in a space curve

The domain of the space curve is the line containing the point 
\((a, b, 0)\) and parallel to \(u\)

The parametric equations for this line are

\[
x = a + t \cos \theta, \hspace{1em} y = b + t \sin \theta, \hspace{1em} z = 0
\]

The space curve has vector equation

\[
r(t) = \langle x(t), y(t), z(t) \rangle, \text{ where}
\]

\[
x(t) = a + t \cos \theta
\]

\[
y(t) = b + t \sin \theta
\]

\[
z(t) = f(a + t \cos \theta, b + t \sin \theta)
\]
We are now ready to compute our derivative!

\[ \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \]

\[ x'(t) = \cos \theta \]
\[ y'(t) = \sin \theta \]
\[ z'(t) = \frac{dz}{dt} = \left( \frac{\partial z}{\partial x} \right) \left( \frac{dx}{dt} \right) + \left( \frac{\partial z}{\partial y} \right) \left( \frac{dy}{dt} \right) \]
\[ = f_x(a + t \cos \theta, b + t \sin \theta) \cos \theta \]
\[ + f_y(a + t \cos \theta, b + t \sin \theta) \sin \theta \]
\[ = \nabla f(a + t \cos \theta, b + t \sin \theta) \cdot \mathbf{u} \]
\[ = D_{\mathbf{u}}(f)(a + t \cos \theta, b + t \sin \theta) \]
The general tangent vector (continued)

- \( \mathbf{r}'(t) = \langle \cos \theta, \sin \theta, D_u(f)(a + t \cos \theta, b + t \sin \theta) \rangle \)

- Note that \( \mathbf{r}'(0) \) is the tangent vector to the curve having vector equation \( \mathbf{r}(t) \) at the point \((a, b, f(a, b))\); it is also a tangent vector to the surface \( z = f(x, y) \) at the point \((a, b, f(a, b))\)

- \( \mathbf{r}'(0) = \langle \cos \theta, \sin \theta, D_u(f)(a, b) \rangle \)

- Compare this to the two special tangent vectors computed earlier

- The general tangent vector is a generalization of those two computations!
Note: subscripts and exponents are not shown here since they do not appear in Winplot

Explicit: $z=f(x,y)=x^2/4+y^2/9$

User-defined functions
- $ff(x,y)=x^2/4+y^2/9$
- $fx(x,y)=x/2$
- $fy(x,y)=2y/9$

Note: Winplot requires user-defined functions to have names of at least two characters, hence the $ff(x,y)$ in place of $f(x,y)$

Animate on $x = a$, $y = b$, $\theta = c$

Unit vector, $u$, translated to $(a, b, 0)$
- Drawn as segment from $(a, b, 0)$ to $(a + \cos(c), b + \sin(c), 0)$
Vertical plane containing \((a, b, 0)\), parallel to unit vector \(\mathbf{u}\)
- Normal vector to vertical plane: \((-\sin(c), \cos(c), 0)\)
- Point: \((a, b, 0)\)

Space curve \(\mathbf{r}(t)\): parametric
\[
\begin{align*}
  x &= a + t \cos(c) \\
  y &= b + t \sin(c) \\
  z &= f(f(a + t \cos(c), b + t \sin(c)))
\end{align*}
\]

Tangent vector to space curve at \((a, b, f(a, b))\)
- Use \(\mathbf{r}'\) values computed earlier; plug in \(t = 0\)
- Segment from \((a, b, f(a, b))\) to \((a + x'(0), b + y'(0), f(a, b) + z'(0))\)
The surface is green, the unit vector is black, the vertical plane is aqua, the curve that is the intersection of the surface and the plane is black, and the tangent vector to the curve at the red point is blue.