Example of a Continuous Function without a Total Derivative at a Point

Consider the following function 
\[
 f(x, y) = \begin{cases} 
  \frac{x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0,0) \\
  0 & \text{if } (x, y) = (0,0)
\end{cases}
\]

This function is continuous since 
\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{r \to 0} \frac{r^3 \cos^2(\theta) \sin(\theta)}{r^2} = \lim_{r \to 0} r \cos^2(\theta) \sin(\theta) = 0
\]
by the squeeze theorem.

Now, \( f_x(0,0) = \lim_{h \to 0} \frac{\frac{h^2}{h^2 + 0^2} - 0}{h} = 0 \) and \( f_y(0,0) = \lim_{h \to 0} \frac{0^2 h}{0^2 + h^2} - 0 \), so that from the quotient rule,
\[
 f_x(x, y) = \begin{cases} 
  \frac{2xy(x^2 + y^2) - 2x(x^2y)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0,0) \\
  0 & \text{if } (x, y) = (0,0)
\end{cases}
\]
and
\[
 f_y(x, y) = \begin{cases} 
  \frac{x^2(x^2 + y^2) - 2y(x^2y)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0,0) \\
  0 & \text{if } (x, y) = (0,0)
\end{cases}
\]
These partial derivatives are not continuous at (0,0) since
\[
\lim_{{(x,y) \to (0,0)}} f_x(x,y) = \lim_{{r \to 0}} \frac{2r^4 \cos(\theta) \sin^3(\theta)}{r^4} = 2\cos(\theta)\sin^3(\theta) \quad \text{and}
\]
\[
\lim_{{(x,y) \to (0,0)}} f_y(x,y) = \lim_{{r \to 0}} \frac{r^4 \cos^2(\theta)\left[\cos^2(\theta) - \sin^2(\theta)\right]}{r^4} = \cos^2(\theta)\left[\cos^2(\theta) - \sin^2(\theta)\right],
\]
and both of these limits depend on the path taken to the origin.

So the guarantee that the function has total derivative, or is differentiable, at the origin is not in effect. As the graph of the function shows there is no unique tangent plane to the surface at the origin.

Now, \( f(h,k) = f(0,0) + f_x(0,0)h + f_y(0,0)k + \varepsilon_1(h,k,0,0)h + \varepsilon_2(h,k,0,0)k \), with

\[
f(h,k) = \frac{h^2k}{h^2 + k^2}
\]

\[
f(0,0) = 0
\]

\[
f_x(0,0)h = 0(h) = 0
\]

\[
f_y(0,0)k = 0(k) = 0
\]

\[
\varepsilon_1(h,k,0,0) = \frac{hk}{h^2 + k^2}
\]

\[
\varepsilon_2(h,k,0,0) = 0
\]

For \( k = 0 \), the \( \lim_{{h \to 0}} \frac{hk}{h^2 + k^2} = \lim_{{h \to 0}} \frac{0}{h^2} = 0 \), and for \( k \neq 0 \), the \( \lim_{{h \to 0}} \frac{hk}{h^2 + k^2} = \frac{0}{k} = 0 \), so

\[
\lim_{{h \to 0}} \varepsilon_1(h,k,0,0) = 0 \quad \text{and} \quad \lim_{{k \to 0}} \varepsilon_2(h,k,0,0) = 0.
\]

However,

\[
\lim_{{(h,k) \to (0,0)}} \varepsilon_1(h,k,0,0) = \lim_{{(h,k) \to (0,0)}} \frac{hk}{h^2 + k^2} = \lim_{{r \to 0}} \frac{\cos(\theta)\sin(2\theta)}{r} = \frac{1}{2},
\]

which can take on any value in the interval \([-\frac{1}{2}, \frac{1}{2}]\) depending on how we approach the origin. Thus, the

\[
\lim_{{(h,k) \to (0,0)}} \varepsilon_1(h,k,0,0) \text{ does not exist.}
\]

For \( (a,b) \neq (0,0) \), a normal to the tangent plane at the point \( (a,b, f(a,b)) = \left(a,b, \frac{a^2b}{a^2+b^2}\right) \) is given by

\[
\left\langle -\frac{2ab^3}{(a^2+b^2)^2}, -\frac{a^2(a^2-b^2)}{(a^2+b^2)^2}, 1 \right\rangle = \left\langle -2\cos(\theta)\sin^3(\theta), -\cos^2(\theta)(\cos^2(\theta) - \sin^2(\theta)), 1 \right\rangle = \left\langle -2\cos(\theta)\sin^3(\theta), -\cos^2(\theta)\cos(2\theta), 1 \right\rangle.
\]
Again the limit of this vector as \((a, b) \to (0,0)\) depends on how we approach the origin. Hence, as \((a, b)\) approaches the origin, there is no well defined normal, and therefore no well defined tangent plane at the origin.