5. (2 points)
A trough of length \( L \) has symmetric parabolic cross sections perpendicular to its length. Each such perpendicular cross section has a vertex a distance \( D \) below the top of the trough and a width across the top of the trough of \( w \). Each differential cross section has a height of \( \sqrt{\frac{y}{2\omega^2}} \) above the top of the trough.

\[ F = \frac{1}{4} \rho g D \left[ \frac{4}{\omega^4} - 1 \right] \frac{dy}{\omega^2} \]

Cross sections: \( y = D \left[ \frac{4}{\omega^4} - 1 \right] \frac{dy}{\omega^2} \) \( - D + \frac{\omega^2}{2} \frac{x^2}{\omega^2} \)

a) If the trough is full of a liquid of mass density \( \rho \) and the acceleration of gravity is \( g \), what is the force due to the liquid acting on one end of the trough?

\[ F = \frac{1}{4} \rho g D \left( \frac{2}{\omega^2} \right) \frac{dy}{\omega^2} \]

b) Calculate the work done in pumping out all of the liquid from the trough to a height \( h \) above the top of the trough.

\[ W = \rho g L D \left[ \frac{4}{\omega^4} - 1 \right] \frac{dy}{\omega^2} \left( h - y \right) \]

\[ W = \rho g L D \left( \frac{4}{\omega^4} - 1 \right) \frac{dy}{\omega^2} \left( h - y \right) \]

6. (9 points)
For each first order differential equation below: Indicate
i. If it is a separable differential equation.
ii. If it is linear differential equation.

Then solve the differential equation subject to the stated initial conditions.

a) \[ \frac{df}{dx} = x^2 + 3x \]

with \( f(-1) = \frac{1}{6} \).

\[ f(x) = \int (x^2 + 3x) \, dx = \frac{x^3}{3} + \frac{3}{2} x^2 + C \]

(i) separable \( \checkmark \)
(ii) linear \( \checkmark \)

\[ f(x) = \frac{x^3}{3} + \frac{3}{2} x^2 - 1 \]

b) \[ \frac{dy}{dx} = 2xy^2 + y^2 \cos(x) \]

with \( y(\pi) = 2 \).

\[ \frac{dy}{x^2 + \sin(x)} \rightarrow \frac{dy}{y^2} = [x + \cos(x)] \, dx \]

(i) separable \( \checkmark \)
(ii) not linear \( \checkmark \)

\[ y(x) = -\frac{1}{x^2 + \sin(x)} + C \]

\[ y(\pi) = \frac{1}{\pi^2 + C} = 2 \]

\[ C + \frac{\pi^2}{2} = -\frac{1}{2} \]

\[ C = -\frac{1}{2} \left( 1 + \pi^2 \right) \]

\[ y(x) = -\frac{1}{x^2 + \sin(x) - 1 + \pi^2} \]
\[ \frac{dy}{dx} = \begin{cases} 
3 & \text{for } x \leq 2 \\
2 & \text{for } x > 2 
\end{cases} \]

\[ g(x) = \begin{cases} 
3 & \text{for } x \leq 2 \\
2 & \text{for } x > 2 
\end{cases} \]

\[ y(x) = \begin{cases} 
3x+1 & \text{for } x \leq 2 \\
\frac{x^2}{2}+x+3 & \text{for } x > 2 
\end{cases} \]

\[ y(x) = \begin{cases} 
3x^2+1 & \text{for } x \leq 2 \\
\frac{x^2}{2}+x+3 & \text{for } x > 2 
\end{cases} \]

\[ g(x) \text{ is continuous at } x = 2 \]

\[ y(x) \text{ must be continuous at } x = 2 \]

\[ y(0) = 3.0 + C_1 = 1 \Rightarrow C_1 = 1 \Rightarrow C_2 = 3 \]

\[ y(x) = \begin{cases} 
3x+1 & \text{for } x \leq 2 \\
\frac{x^2}{2}+x+3 & \text{for } x > 2 
\end{cases} \]

\[ y(x) = \begin{cases} 
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x \leq 2 \\
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x > 2 
\end{cases} \]

\[ y(x) = \begin{cases} 
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x \leq 2 \\
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x > 2 
\end{cases} \]

\[ y(x) = \begin{cases} 
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x \leq 2 \\
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x > 2 
\end{cases} \]

\[ y(x) = \begin{cases} 
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x \leq 2 \\
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x > 2 
\end{cases} \]

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\end{cases} \]

\[ y(x) = \begin{cases} 
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x \leq 2 \\
\frac{3}{4}x^2 + \frac{3}{20}x^2 & \text{for } x > 2 
\end{cases} \]
Problem 10

Exponential Model:
k = 0.15, initial pop = 300

Logistic Model:
k = 0.15, initial pop = 300

Problem 7

Slope Fields

\[ \dot{x} = 1; \quad \dot{y} = x - 2xy \]
7. (3 points)  

a) Solve \( \frac{dy}{dx} + 2xy = x \) subject to \( y(0) = a \).

\[
\frac{dy}{dx} + 2xy = x \implies e^{-2x} \frac{dy}{dx} + 2xe^{-2x}y = xe^{-2x}
\]

Integrating factor: \( e^{\int 2x \, dx} = e^{x^2} \)

\[
y(x) = \frac{1}{a} + (a - \frac{1}{2}) e^{-x^2} + \frac{1}{a} \int (-e^{-x^2}) + ce^{-x^2} \, dx
\]

b) Generate and attach a graph of the direction field graph for this equation with a window from \(-2.5 \text{ to } 2.5\) in both the \( x \) and \( y \) directions. On this same graph plot the five solutions \( y(x) \) for \( a = -0.5, 0, a = 0.5, a = 1 \) and \( a = 1.5 \).

In WinPlot choose the 2-dim Window and in the Equa menu pick DEq / dy/dt and set \( x' = 1 \) and \( y' = f(x, y) \). Here \( f(x, y) \) is the right hand side of the first order ODE \( y' = f(x, y) \). Choose the vectors checkbox and press [ok] to display the direction field (with arrows indicating the flow direction)!

From the View menu choose View/ Set corners to set the relevant window for \( x \) and \( y \). From the One menu select dy/dt trajectory. Then set the initial conditions at \( x = 0 \) by leaving both \( x \) and \( t \) set at \( 0 \) and entering the desired value of \( y \) at \( x = 0 \) (i.e., \( a \)). The option marked both should be checked and the trajectory is displayed by pressing either the [draw] or [watch] buttons.

c) Explain what happens at \( a = 0.5 \). When \( a = \frac{1}{2} \), \( y(x) = \frac{1}{2} \) is the solution.

\[
\lim_{|x| \to \infty} y(x) = \frac{1}{2} t \ln (a - \frac{1}{2}) + \frac{1}{2} = \frac{1}{2} \ln (a - \frac{1}{2}) + \frac{1}{2}
\]

so all solutions approach the solution for \( a = \frac{1}{2} \) as \( |x| \to \infty \).

8. (2 points) Linear Circuits:

a) A resistor \( R \) (5.0 \( \Omega \)) and an inductor \( L \) (0.150 H) are hooked up in series to a power supply \( V \) (5.0 volts) which establishes a steady state current of \( \frac{V}{R} \) (1.0 amperes). The power supply is then removed and the current decays in a manner described by:

\[
R \frac{di}{dt} + L \frac{di}{dt} = 0 \rightarrow \frac{di}{dt} = -\frac{R}{L} \frac{di}{dt} \rightarrow L \frac{di}{dt} = -\frac{R}{L} i(t) \rightarrow i(t) = \frac{V}{R} e^{-\frac{R}{L} t} \rightarrow \ln |i(t)| = -\frac{R}{L} t + C
\]

\[
t = \frac{1}{2} (\ln (\frac{V}{R})) = \frac{\ln(4)}{2 \pi} = 0.04157 \text{ sec}
\]

b) A capacitor \( C \) (2.0 \( \mu F \)) is charged to a potential difference \( V \) (10.0 volts) . The power supply is removed and the capacitor is allowed to discharge through a resistor \( R \) (1.5M\( \Omega \)) . The charge \( Q \) on the capacitor is given by \( Q = CV \) and decays in a manner described by:

\[
R \frac{dQ}{dt} + \frac{Q}{C} = 0 \rightarrow \frac{dQ}{dt} = -\frac{Q}{C} \frac{Q(t)}{Q_o} \rightarrow \ln \left( \frac{Q(t)}{Q_o} \right) = -\frac{t}{RC}
\]

\[
t = RC \ln \left( \frac{Q(t)}{Q_o} \right) = 4.1588 \text{ sec}
\]

For arbitrary values of \( R, L \) and \( V \), how long after the power supply is removed is the current down to \( \frac{V}{4R} \)?

For arbitrary values of \( R, C \) and \( V \) (the power supply voltage), how long after the power supply is removed is the charge down to \( \frac{CV}{4} \)?
9. (2 points) Concentration Dilution.

a) A tank of volume $V$ has a chlorine concentration of $c_0$. It is desired to reduce this to $c_f$ by pumping in water with a chlorine concentration of $c_I$ ($c_0 > c_f > c_I$). To prevent over flowing water is pumped out at the same rate. If the water is pumped in at a rate of $r$ gallons per minute and if the water is so well mixed that at any time the chlorine concentration is uniform throughout the tank, how long will it take to reach the desired concentration?

$$\frac{dA}{dt} = \frac{c_I - c}{V} A(t)$$
$$\frac{dc}{dt} = \frac{r}{V} (c_I - c)$$

Pure water: $c(t) = 0$, $t = \frac{V}{r} \ln \left( \frac{c_0 - c_I}{c_I - c_f} \right)$

b) How long would it take if 'pure' water were used instead of the water with a chlorine concentration of $c_I$? $c(t) = c_0 e^{-\frac{t}{V}}$

10. (6 points) Population Growth.

a) The most naive (and most frightening) model for population growth is the exponential growth model. This assumes that the rate of growth is directly proportional to the current population size. In symbols this is expressed as $\frac{dN}{dt} = kN$, where $N$ is the number of individuals in the population at time $t$ and $k$ is a constant. Roughly, $k$ is the number of new 'offspring' produced by each individual in a relevant unit of time. Solve $\frac{dN}{dt} = kN$, subject to the initial condition that $N(0) = N_0$.

$$\frac{dN}{dt} = kN \Rightarrow \ln N = kt + C \Rightarrow N(t) = N_0 e^{kt}$$

b) The exponential model is obviously unrealistic for large times in that it leads to unbounded population growth. There is only so much matter in the universe! At some point limited food and living space limit the population size of any species. One way to model this behavior mathematically is to modify the rate equation as follows: $\frac{dN}{dt} = kN(1 - \frac{N}{L})$, where $L$ is the limiting population size. Initially $N$ increases exponentially as before; however, as $N$ approaches $L$ the rate of growth slows to zero. Hence, $N = L$ is a horizontal asymptote of the solution. This rate equation is sometimes called the logistic equation.

For the logistic equation model what value of $N$ makes the rate of growth a maximum?

$$\frac{dN}{dt} = kN(1 - \frac{N}{L}) \Rightarrow \text{Rate of growth} = kN \left( 1 - \frac{N}{L} \right)$$

$$\frac{dN}{dt} = 0 \Rightarrow \frac{N}{L} = \frac{1}{2}$$

(c) What is this maximum rate of growth? How does this compare to the rate of growth of the exponential model for the same population size?

$$\text{Logistic rate of growth} = \frac{kL}{2}$$

$$\text{Exponential rate of growth} = \frac{kL}{2}$$

(d) Solve the logistic equation, subject to the initial condition that $N(0) = N_0$.

$$\frac{dN}{dt} = kN(1 - \frac{N}{L})$$

$$\int \frac{dN}{N(L-N)} = k \Rightarrow \ln \left( \frac{N(t)}{L-N(t)} \right) = \frac{kL}{N_0} \Rightarrow N(t) = \frac{N_0 e^{kt}}{1 + e^{kt} \frac{N(t)}{L}}$$

(e) Evaluate $\lim_{t \to \infty} N(t) = L$

$$N(t) = \frac{L}{1 + \frac{N_0}{L} e^{-kt}} \Rightarrow \lim_{t \to \infty} N(t) = L$$

(f) For $N_0 = 300$, $L = 500,000$ and $k = 0.15$ year$^{-1}$, according to the exponential model how long would it take the population to double to 600? To increase to 100,000?

$$t = \frac{\ln(2)}{k} = 4.62 \text{ years}$$

$$t = \frac{\ln(100,000)}{k}$$

Doubling time = \frac{\ln(2)}{k} = 4.62 \text{ years}$$

$$t = \frac{\ln(100,000)}{k}$$

Doubling time = \frac{\ln(2)}{k} = 4.62 \text{ years}$$
g) For \( N_0 = 300 \), \( L = 500,000 \) and \( k = 0.15 \text{ year}^{-1} \), according to the logistic model how long would it take the population to double to 600? To increase to 100,000? 

\[
t = \frac{1}{k} \ln \left( \frac{N(t) - L - N_0}{N_0} \right) \text{ time to increase to } 100,000
\]
\[
t = \frac{1}{k} \ln \left( \frac{N(t) - L - N_0}{N_0} \right) \text{ doubling time}
\]

\[
t = 4.624 \text{ years}
\]

h) For \( N_0 = 300 \), \( L = 500,000 \) and \( k = 0.15 \text{ year}^{-1} \), make a careful graph of the solutions of both the exponential and logistic models. A graphing calculator or computer program would be helpful here.

---

Self-Assessment: (2 bonus points)

a) Describe three strengths in your performance on this project. Include why each is a strength.

\[
\frac{N(t)}{L - N(t)} = \frac{N_0 e^{kt}}{N_0} = \left( L - N(t) \right) \left[ \frac{N_0 e^{kt}}{L - N(t)} \right] = N(t) \Rightarrow N(t) \left[ 1 + \frac{N_0 e^{kt}}{L - N(t)} \right] = \frac{N_0 e^{kt}}{L - N(t)}
\]

b) What are three things that could be improved about your performance on this project. Explain specifically how you will make these improvements.

\[
e^{kt} = \frac{N(t)}{N_0} \cdot \frac{L - N_0}{L - N(t)}
\]

\[
k t = \ln \left( \frac{N(t)}{N_0} \cdot \frac{L - N_0}{L - N(t)} \right)
\]

c) Identify two things about this project which are still unclear to you.

\[
t = \frac{1}{k} \ln \left( \frac{N(t)}{N_0} \cdot \frac{L - N_0}{L - N(t)} \right)
\]

d) Identify two insights that you have acquired in doing this project.
### Problem 10

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<th>Population Logistical Model</th>
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#### Population Growth

- **Population Exponential Model**
- **Population Logistical Model**

![Population Growth Graph](image-url)